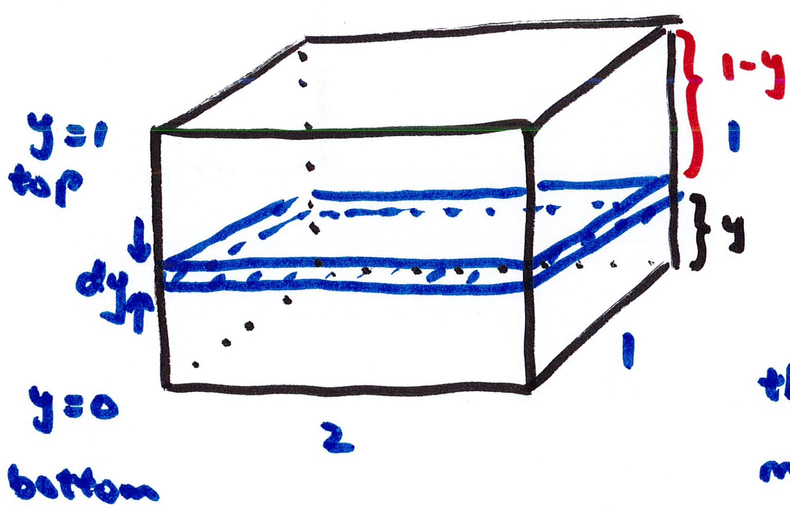
6.7 Physical Applications (part 2)

work problem

work to pump out water from a tank.

example An aquarium length 2 m width 1 m height 1 m is full of water.

Find the amount of work done to pump out all the water over the top.



just like with the chain problem, let's consider the amount of work to pump out one "slice" of the water.

thickness = dy

mass of slice = length · width · thickness · density

$= 2 \cdot 1 \cdot dy = 2dy \cdot \rho$

slice is at height of y from bottom

it needs to go a distance of $1-y$ to get out

work to move this slice = force · distance

force = weight

$= (2\rho\, dy)(g)(1-y) = 2\rho g (1-y)\, dy$

mass   gravity   dist. to go

accumulate all the water to move: from $y=0$ to $y=1$

$$W = \int_0^1 2\rho g (1-y)\, dy = \cdots = \boxed{\rho g} \ (J)$$

ρ for water 1000 kg/m^3

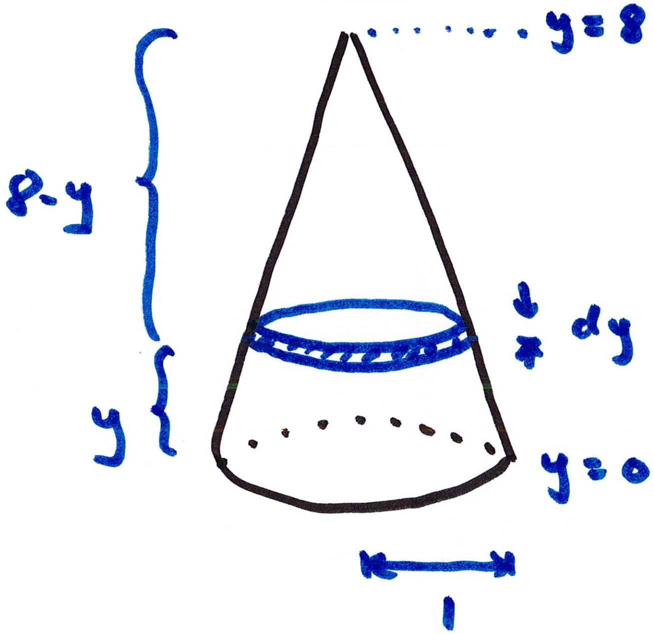
$$= 2\rho g \int_0^1 (1-y)\, dy$$

$$= 2\rho g \left(y - \tfrac{1}{2}y^2\right)\Big|_0^1 = 2\rho g = \left(1 - \tfrac{1}{2}\right) = \rho g$$

example The tank is in the shape of a cone vertex up.

The height is 8 m and the radius at the base is 1 m.

The tank is full of water, how much work to pump all the water out? What if the tank is half full?



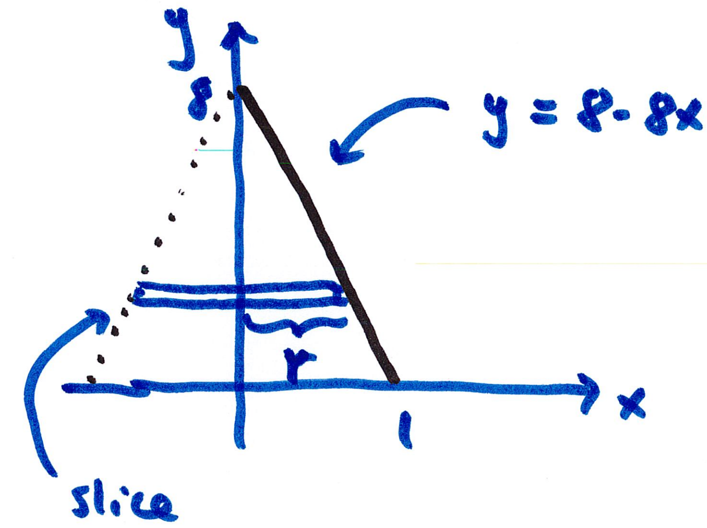
find the work to move one slice

radius of slice?

notice it depends on height (y)

radius = some function of y

here are two ways to get that :



cross section on xy axes

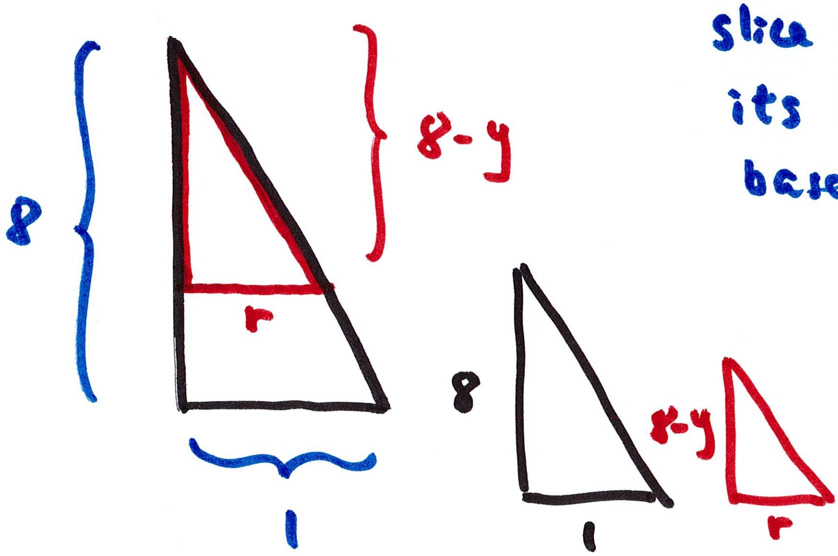
note the radius r is the same as x

$y = 8 - 8x$ solve for x

$8x = 8 - y$

$x = 1 - \frac{1}{8}y = r$

the other way : Similar triangles



slice is at base of red triangle

its height is the dist. to go : 8-y

base is radius

$$\frac{1}{8} = \frac{r}{8-y}$$

Similar : same shape

ratio of corresponding size is equal

$$r = \frac{1}{8}(8-y)$$

$$= 1 - \frac{1}{8}y$$

weight of slice: $\pi\underbrace{(1-\frac{1}{8}y)^2}_{\text{radius}}\cdot\underbrace{\rho}_{\text{density}}\cdot g\cdot \underbrace{dy}_{\text{thickness}}$

it needs to go $8-y$ to top

work $=\pi(1-\frac{1}{8}y)^2\rho g\, dy\,(8-y)$

$=\rho g\pi(1-\frac{1}{8}y)^2(8-y)\,dy$

to pump full tank out: $W=\int_0^8 \rho g\pi(1-\frac{1}{8}y)^2(8-y)\,dy=\cdots=\boxed{16\pi\rho g}$

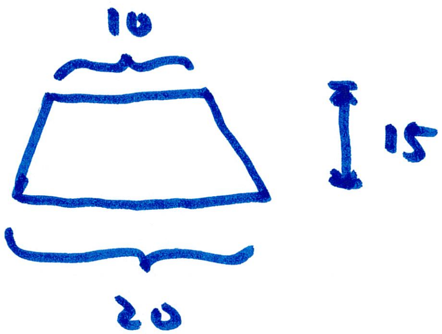
$=156{,}800\pi$

if tank is partially half empty

$$W=\int_0^4 \rho g\pi(1-\frac{1}{8}y)^2(8-y)\,dy=\cdots=147{,}000\pi$$

example (hydrostatic pressure/force)

A small dam is in the shape of a trapezoid



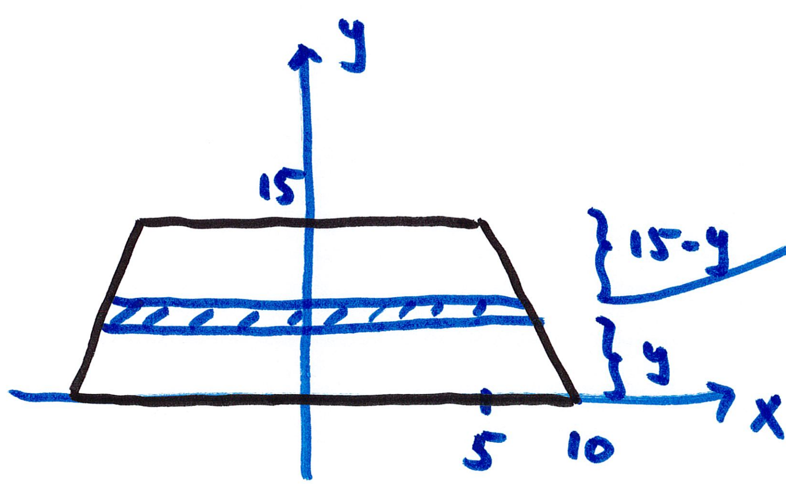
all in meters

The water level is even with the top.

what is the force on this dam?

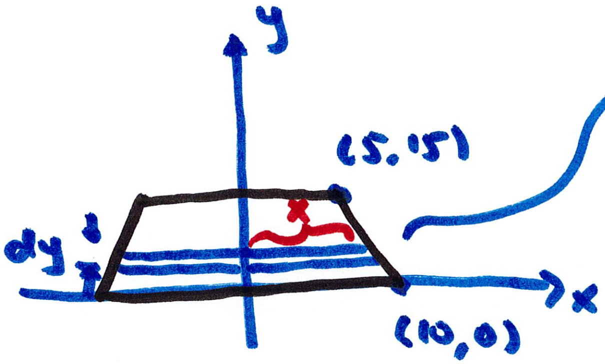
hydrostatic pressure = density · gravity · depth

force = pressure · area



one strip of the dam
at height of y
15-y from top where water level is
so it is 15-y deep from surface

notice length is a function
of y



$y = -3x + 30$

or $x = -\frac{1}{3}y + 10$ → half of length of strip

$\text{pressure} = \rho g(15-y)$

$\text{area of one strip} = 2\left(-\frac{1}{3}y+10\right)dy$

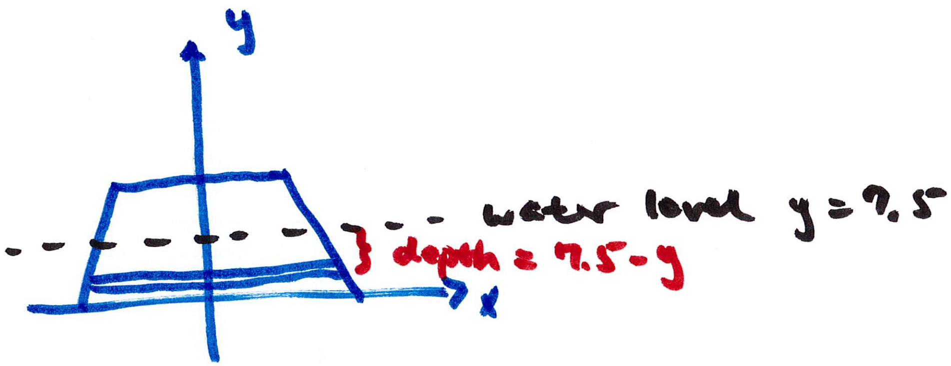
$\text{force on one strip} = \rho g(15-y)(2)\left(-\frac{1}{3}y+10\right)dy$

$= 2\rho g(15-y)\left(-\frac{1}{3}y+10\right)dy$

accumulate over portion of dam submerged $y=0 \to y=15$

$$\int_0^{15} 2\rho g(15-y)\left(-\frac{1}{3}y+10\right)dy = \cdots = \boxed{1875\rho g} \quad (N)$$

If the dam is only half submerged, then in addition to the upper bound of integration, the "depth" also changes, since there is no hydrostatic pressure/force for the portion NOT submerged



force on the dam:

$$\int_0^{7.5} 2\rho g(7.5-y)\left(-\frac{1}{3}y+10\right)dy = \cdots = 515.625\rho g \quad (N)$$