8.3 Trig Integrals (part 1)

today: focus on sine and cosine

next time: tangent and secant

Substitution involving $\sin x$ and $\cos x$ → they are derivatives of each other
(related to)

$$\int \frac{\sin x}{\cos x}\,dx$$

$u = \cos x$

$du = -\sin x\,dx$

$$= \int \frac{1}{u} \cdot -du = -\int \frac{1}{u}\,du = -\ln|u| + C = -\ln|\cos x| + C$$

example

$$\int \frac{1}{1-\sin x}\,dx$$

need to do something else first

$$= \int \frac{1}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x}\,dx$$

$$= \int \frac{1+\sin x}{1-\sin^2 x}\,dx \qquad \sin^2 x + \cos^2 x = 1$$

$$= \int \frac{1+\sin x}{\cos^2 x}\,dx = \int \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\,dx$$

$$= \int \frac{1}{\cos^2 x}\,dx + \underbrace{\int \frac{\sin x}{\cos^2 x}\,dx}_{u = \cos x \quad du = -\sin x\,dx}$$

$$= \int \sec^2 x\,dx + \int \frac{-1}{u^2}\,du \qquad -\int u^{-2}\,du = -\left(\frac{u^{-1}}{-1}\right) + c$$

$$= \tan x + \frac{1}{u} + C = \boxed{\tan x + \frac{1}{\cos x} + C}$$

<u>example</u>

$$\int_{-\frac{\pi}{2}}^{0} \sqrt{1+\cos(2x)}\, dx$$

do something similar under the root

$$\sqrt{\frac{1+\cos(2x)}{1} \cdot \frac{1-\cos(2x)}{1-\cos(2x)}} = \sqrt{\frac{1-\cos^2(2x)}{1-\cos(2x)}}$$

$$= \sqrt{\frac{\sin^2(2x)}{1-\cos(2x)}} = \frac{|\sin(2x)|}{\sqrt{1-\cos(2x)}}$$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$|\sin(2x)| = \begin{cases} \sin(2x) & \text{if } \sin(2x) \geq 0 \\ -\sin(2x) & \text{if } \sin(2x) < 0 \end{cases}$$

integration limits: $-\frac{\pi}{2} \leq x \leq 0$

over this interval, $\sin(2x) < 0$

since $\sin(2x) < 0$, $|\sin(2x)| = -\sin(2x)$

so integral becomes

$$\cancel{\int_{-\frac{\pi}{2}}^{0} \frac{-\sin(2x)}{\sqrt{1+\cos(2x)}}\,dx} \qquad \int_{-\frac{\pi}{2}}^{0} \frac{-\sin(2x)}{\sqrt{1-\cos(2x)}}\,dx$$

$u = 1-\cos(2x)$

$du = 2\sin(2x)\,dx$

$\sin(2x)\,dx = \frac{1}{2}du$

$-\sin(2x)\,dx = -\frac{1}{2}du$

$x=0 \to u = 1-\cos(0) = 0$

$x=-\frac{\pi}{2} \to u = 1-\cos(-\pi) = 2$

$$= \int_{2}^{0} \frac{-1}{2u^{1/2}}\,du = -\frac{1}{2}\int_{2}^{0} u^{-1/2}\,du$$

$$= -\frac{1}{2}\left(2u^{1/2}\right)\Big|_{2}^{0} = -u^{1/2}\Big|_{2}^{0} = 0 - -2^{1/2} = \boxed{\sqrt{2}}$$

example

$$\int \sin^2 x \cos^5 x \, dx$$

basic idea: use $\cos^2 x + \sin^2 x = 1$ then substitution

$$= \int \sin^2 x \cdot \underbrace{\cos^4 x}_{(\cos^2 x)^2} \cdot \underbrace{\cos x \, dx}_{\text{goes into } du \text{ if } u = \sin x}$$

$$= \int \underbrace{\sin^2 x \,(1-\sin^2 x)^2}_{\text{let } u = \sin x} \cdot \underbrace{\cos x \, dx}_{du = \cos x \, dx}$$

$$= \int u^2 (1-u^2)^2 \, du = \int u^2 (1 - 2u^2 + u^4) \, du$$

$$= \int u^2 - 2u^4 + u^6 \, du = \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C$$

$$= \boxed{\frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C}$$

Strategy for $\int \sin^m x \cos^n x \, dx$

case 1: if m or n is <u>positive and odd</u> and the other is any real number

Split one factor of the part with odd power then use $\sin^2 x + \cos^2 x = 1$ and turn everything else into the other one then do substitution

case 2: if m and n are both even and nonnegative

use the half-angle identities

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

example

$$\int \sin^2 x \cos^2 x \, dx$$

$$= \int \frac{1-\cos(2x)}{2} \cdot \frac{1+\cos(2x)}{2} \, dx$$

$$= \int \frac{1-\cos^2(2x)}{4} \, dx$$

$$= \frac{1}{4} \int 1-\cos^2(2x) \, dx = \frac{1}{4} \int \sin^2(2x) \, dx$$

use $\sin^2(\square) = \dfrac{1-\cos(2\square)}{2}$

$$= \frac{1}{4} \int \frac{1-\cos(4x)}{2} \, dx = \frac{1}{8} \int 1-\cos(4x) \, dx$$

$$= \boxed{\frac{1}{8}\left(x - \frac{1}{4}\sin 4x\right) + C}$$

example

$$\int \cos^{-3/2}x \sin^3 x \, dx$$

case 1: sine has positive and odd power

save one factor, then turn remainder into cosine

$$\int \cos^{-3/2}x \cdot \underbrace{\sin^2 x}_{\text{use } \sin^2 x + \cos^2 x = 1} \cdot \sin x \, dx$$

$$= \int \cos^{-3/2}x \cdot (1 - \cos^2 x) \cdot \sin x \, dx$$

now let $u = \cos x$

$du = -\sin x \, dx$

$$= \int u^{-3/2}(1-u^2) \cdot -du = -\int \left(u^{-3/2} - u^{1/2}\right) du$$

$$= -\left(-2u^{-1/2} - \frac{2}{3}u^{3/2}\right) + C = 2u^{-1/2} + \frac{2}{3}u^{3/2} + C$$

$$= \boxed{2\cos^{-1/2}x + \frac{2}{3}\cos^{3/2}x + C}$$

Integrals involving tangent and secant

Similar to with sine and cosine, we want to remember how their derivatives are related

we know : $\frac{d}{dx} \tan x = \sec x$

$\frac{d}{dx} \sec x = \sec x \tan x$

and $\tan^2 x + 1 = \sec^2 x$

(you can get this by dividing $\sin^2 x + \cos^2 x = 1$ through by $\cos^2 x$)

<u>example</u> $\int \tan^3 x \, dx$

we know derivative of $\tan x$ is $\sec^2 x$, so try to bring in $\sec^2 x$

$\int \tan x \cdot \underbrace{\tan^2 x}\, dx$

we can find $\sec^2 x$ in here : $\tan^2 x + 1 = \sec^2 x$

$\tan^2 x = \sec^2 x - 1$

$= \int \tan x \cdot (\sec^2 x - 1) \, dx$

$$= \int \left(\tan x \cdot \sec^2 x - \tan x\right) dx$$

$$= \underbrace{\int \tan x \cdot \sec^2 x \, dx}_{\text{Substitution}} - \underbrace{\int \tan x \, dx}_{\text{rewrite}}$$

Substitution

$u = \tan x$

$du = \sec^2 x \, dx$

rewrite: $\tan x = \dfrac{\sin x}{\cos x}$ then Substitution with $U = \cos x$

$dU = -\sin x \, dx$

$$= \ldots = \boxed{\frac{1}{2}\tan^2 x + \ln|\cos x| + C}$$

Same idea w/ integrals involving $\cot x$ and $\csc x$

$$\frac{d}{dx} \cot x = -\csc^2 x$$

$$\frac{d}{dx} \csc x = -\csc x \cot x$$

$\sin^2 x + \cos^2 x = 1$

and $\quad 1 + \cot^2 x = \csc^2 x \quad$ (get this by dividing by $\sin^2 x$)

<u>example</u> $\quad \int \cot^2 x \, dx$

$$= \int (\csc^2 x - 1)\, dx$$

$$= \int \csc^2 x \, dx - \int 1 \cdot dx$$

$$= -\cot x - x + C$$

because $\frac{d}{dx} \cot x = -\csc x \rightarrow \frac{d}{dx}(-\cot x) = \csc x$

so $\int \csc x \, dx = -\cot x + C$

<u>example</u> $\int \sec^4 x \, dx$

$$= \int \sec^2 x \sec^2 x \, dx$$

$$= \int \sec^2 x \cdot (1 + \tan^2 x) \, dx$$

$$= \int \sec^2 x \, dx + \underbrace{\int \sec^2 x \tan^2 x \, dx}$$

$u = \tan x$

$du = \sec^2 x \, dx$

$$= \int \sec^2 x \, dx + \int u^2 \, du$$

$$= \tan x + \frac{1}{3} u^3 + C = \tan x + \frac{1}{3} \tan^3 x + C$$