8.3 Trig Integrals (continued)

example $\int \tan^3 x \, dx$

just like with sin x and cos x, try to bring in derivative of tan x or sec x to form du

$\frac{d}{dx} \tan x = \sec^2 x$

$\frac{d}{dx} \sec x = \sec x \tan x$

useful identity: $\tan^2 x + 1 = \sec^2 x$

two ways:

$$\int \tan^3 x \, dx$$

$$= \int \tan x \cdot \tan^2 x \, dx$$

$$= \int \tan x \cdot (\sec^2 x - 1) \, dx$$

$$= \underbrace{\int \tan x \cdot \sec^2 x \, dx}_{\substack{u = \tan x \\ du = \sec^2 x \, dx}} - \underbrace{\int \tan x \, dx}_{\substack{\int \frac{\sin x}{\cos x} dx \\ U = \cos x \\ dU = -\sin x \, dx}}$$

$$= \frac{1}{2}\tan^2 x + \ln|\cos x| + C$$

see notes from last time for remaining steps

Second way

$$\int \tan^3 x \, dx$$

$$= \int \frac{\tan^3 x}{\sec x} \cdot \sec x \, dx = \int \underbrace{\frac{\tan^2 x}{\sec x}}_{\text{rest of this needs to turn into } \sec x} \underbrace{\tan x \sec x \, dx}_{du \text{ if } u = \sec x}$$

rest of this needs to turn into sec x

$$\tan^2 x + 1 = \sec^2 x$$

$$= \int \frac{\sec^2 x - 1}{\sec x} \cdot \sec x \tan x \, dx \qquad u = \sec x$$
$$du = \sec x \tan x \, dx$$

$$= \int \frac{u^2 - 1}{u} \cdot du = \int \left(u - \frac{1}{u}\right) du = \frac{1}{2} u^2 - \ln|u| + C$$

$$= \boxed{\frac{1}{2} \sec^2 x - \ln|\sec x| + C}$$

example $\displaystyle\int \frac{1}{\sec x - 1}\,dx$

another way to handle $\tan x$ and $\sec x$: turn into $\cos x$ and $\sin x$

$$= \int \frac{1}{\frac{1}{\cos x} - 1}\,dx = \int \frac{1}{\frac{1}{\cos x} - 1} \cdot \frac{\cos x}{\cos x}\,dx$$

$$= \int \frac{\cos x}{1-\cos x}\,dx = \int \frac{\cos x}{1-\cos x} \cdot \frac{1+\cos x}{1+\cos x}\,dx$$

$$= \int \frac{\cos x + \cos^2 x}{1-\cos^2 x}\,dx = \int \frac{\cos x + \cos^2 x}{\sin^2 x}\,dx$$

$$= \underbrace{\int \frac{\cos x}{\sin^2 x}\,dx}_{\substack{u = \sin x \\ du = \cos x\,dx \\ \vdots}} + \int \frac{\cos^2 x}{\sin^2 x}\,dx = \int \frac{\cos x}{\sin^2 x}\,dx + \int \frac{1-\sin^2 x}{\sin^2 x}\,dx$$

$$= \int \frac{\cos x}{\sin^2 x}\,dx + \int \frac{1}{\sin^2 x}\,dx - \int \frac{\sin^2 x}{\sin^2 x}\,dx$$

$$= \underbrace{\int \frac{\cos x}{\sin^2 x}\,dx}_{\substack{u = \sin x \\ du = \cos x\,dx}} + \underbrace{\int \csc^2 dx}_{-\cot x} - \underbrace{\int 1\,dx}_{\text{easy}}$$

$$\vdots$$

$$= \boxed{-\frac{1}{\sin x} - \cot x - x + C}$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\frac{d}{dx} \cot x = -\csc^2 x$$

$$\int \csc^2 x\,dx = -\cot x + C$$

Strategy for $\int \tan^m x \sec^n x \, dx$

if **n is even and positive** → save $\sec^2 x$ then use identity and substitution

(sec x has positive even power)

if **m is odd and positive** → save sec x tan x then identity and substitution

(tan x has odd positive power)

example $\int \tan x \sec^7 x \, dx$

tan x has odd power, so save tan x sec x and use u = sec x

$$= \int \sec^6 x \cdot \sec x \tan x \, dx$$

$u = \sec x \quad du = \sec x \tan x \, dx$

$$= \int u^6 \, du = \frac{1}{7} u^7 + C = \boxed{\frac{1}{7} \sec^7 x + C}$$

what if we went against the suggestion and saved $\sec^2 x$ instead?

$u = \tan x$

$du = \sec^2 x \, dx$

$$\int \tan x \sec^7 x \, dx$$

$$= \int \underbrace{\tan x}_{u} \cdot \underbrace{\sec^5 x}_{?} \cdot \underbrace{\sec^2 x \, dx}_{du}$$

odd power so $\tan^2 x + 1 = \sec^2 x$

cannot completely get rid of $\sec x$

this is a sign that you used the wrong u

<u>example</u> $\int \tan^5 x \sec^6 x \, dx$

tanx has odd and sec x has even so we can save either $\sec^2 x$ or $\sec x \tan x$

try saving $\sec^2 x \to u = \tan x$

$$\int \underbrace{\tan^5 x}_{u^5} \sec^4 x \underbrace{\sec^2 x \, dx}_{du}$$

$$\hookrightarrow (\sec^2 x)^2 = (\tan^2 x + 1)^2 = (u^2+1)^2$$

$$= \int u^5 (u^2+1)^2 du = \int u^5 (u^4 + 2u^2 + 1) du$$

$$= \int u^9 + 2u^7 + u^5 \, du = \frac{1}{10} u^{10} + \frac{1}{4} u^8 + \frac{1}{6} u^6 + C$$

$$= \boxed{\frac{1}{10} \tan^{10} x + \frac{1}{4} \tan^8 x + \frac{1}{6} \tan^6 x + C}$$

try saving $\sec x \tan x \to u = \sec x$

$$\int \tan^4 x \underbrace{\sec^5 x}_{u^5} \underbrace{\sec x \tan x \, dx}_{du}$$

$$\hookrightarrow (\tan^2 x)^2 = (\sec^2 x - 1)^2 = (u^2-1)^2$$

$$\int (u^2-1)^2 u^5 \, du = \int (u^4-2u^2+1)(u^5)\, du$$

$$= \int u^9 - 2u^7 + u^5 \, du = \frac{1}{10}u^{10} - \frac{1}{4}u^8 + \frac{1}{6}u^6 + C$$

$$= \boxed{\frac{1}{10}\sec^{10}x - \frac{1}{4}\sec^8 x + \frac{1}{6}\sec^6 x + C}$$