8.4 Trigonometric Substitution (part 1)

$$\int \frac{1}{\sqrt{1-x^2}}\,dx$$

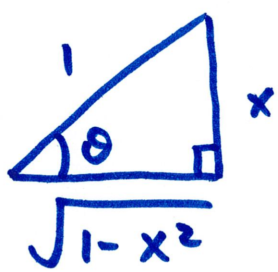
u subs doesn't work: $u = 1-x^2$

$du = -2\underline{\underline{x}}\,dx$

not in integral

trig sub can handle this

consider the square root part $\sqrt{1-x^2}$ as one side of a right-angle triangle



from the triangle, relate x and θ in the simplest way possible

$\sin\theta = \frac{x}{1} \rightarrow x = \sin\theta$

$dx = \cos\theta\, d\theta$

sub these into integral

eliminate all x

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-\sin^2\theta}} \cos\theta \, d\theta = \int \frac{1}{\sqrt{\cos^2\theta}} \cos\theta \, d\theta$$

$$= \int \frac{1}{\cos\theta} \cos\theta \, d\theta = \int d\theta = \theta + C$$ put this in terms of x

from $x = \sin\theta \rightarrow \theta = \sin^{-1}(x)$

so, $\theta + C = \boxed{\sin^{-1}(x) + C}$

example

$$\int \frac{x^3}{\sqrt{x^2+4}}\,dx$$

how to draw the triangle?

from the radical part $\sqrt{x^2+4}$

square root

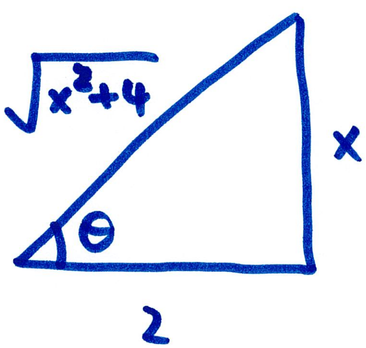
draw a triangle with sides: $\sqrt{x^2+4}$, x, 2

square root

decide which is hypotenuse: $\sqrt{x^2+4}$ because ~~it is the~~ its square is the sum of squares of the other two

we can put the remaining two however we want

BUT, if there is a constant we normally put as adjacent



relate x and θ in simplest way possible

here, $\tan\theta = \frac{x}{2} \rightarrow x = 2\tan\theta$

$$dx = 2\sec^2\theta \, d\theta$$

Sub into integral, eliminate x and dx

$$\int \frac{x^3}{\sqrt{x^2+4}}\,dx = \int \frac{(2\tan\theta)^3}{\sqrt{\underbrace{4\tan^2\theta+4}}}\, 2\sec^2\theta\, d\theta$$

$$4\tan^2\theta+4 = 4(\tan^2\theta+1) = 4\sec^2\theta$$

$$= \int \frac{8\tan^3\theta}{\sqrt{4\sec^2\theta}}\, 2\sec^2\theta\, d\theta = \int \frac{8\tan^3\theta}{2\sec\theta}\cdot 2\sec^2\theta\, d\theta$$

$$= 8\int \tan^3\theta \sec\theta\, d\theta = 8\int \tan^2\theta \sec\theta\tan\theta\, d\theta$$

$$= 8\int (\sec^2\theta-1)\sec\theta\tan\theta\, d\theta = 8\int (u^2-1)\,du$$

$u = \sec\theta$

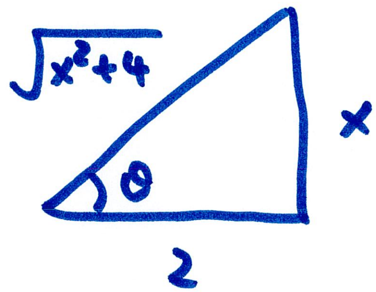
$du = \sec\theta\tan\theta\, d\theta$

$$= 8\left(\frac{u^3}{3} - u\right) + C$$

$$= \frac{8}{3}\sec^3\theta - 8\sec\theta + C$$

but we want this in terms of x

bring back triangle:



$$\sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$= \frac{\sqrt{x^2+4}}{2}$$

$$= \frac{8}{3}\left(\frac{\sqrt{x^2+4}}{2}\right)^3 - 8\left(\frac{\sqrt{x^2+4}}{2}\right) + C$$

$$= \boxed{\frac{1}{3}(x^2+4)^{3/2} - 4(x^2+4)^{1/2} + C}$$

example

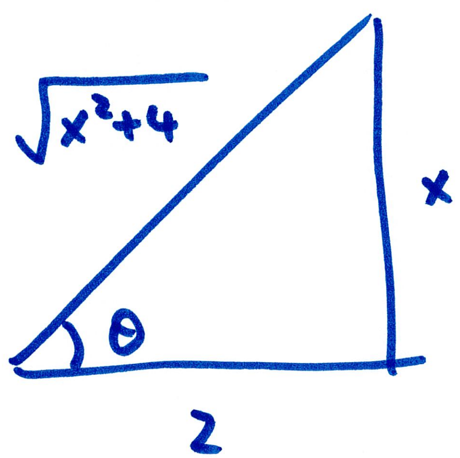
$$\int_0^2 \frac{x^2}{x^2+4}\,dx$$

$$= \int_0^2 \frac{x^2}{\left(\sqrt{x^2+4}\right)^2}\,dx$$

now draw triangle with sides: $\sqrt{x^2+4}$, x, 2

hypotenuse : $\sqrt{x^2+4}$ (square is sum of squares of others)

2 is constant, put as adjacent



$\tan\theta = \frac{x}{2} \longrightarrow \theta = \tan^{-1}\left(\frac{x}{2}\right)$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

now change the bounds

upper : $x = 2 \rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}$

lower : $x = 0 \rightarrow \theta = \tan^{-1}(0) = 0$

$$\int_0^{\pi/4} \frac{(2\tan\theta)^2}{(2\tan\theta)^2+4}\, 2\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \frac{4\tan^2\theta}{4\tan^2\theta+4} \cdot 2\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \frac{4\tan^2\theta}{4\underbrace{(\tan^2\theta+1)}_{\sec^2\theta}} \cdot 2\sec^2\theta\, d\theta \quad = \int_0^{\pi/4} 2\tan^2\theta\, d\theta$$

$$= 2\int_0^{\pi/4} (\sec^2\theta - 1)\, d\theta = 2\int_0^{\pi/4} \sec^2\theta\, d\theta - 2\int_0^{\pi/4} d\theta$$

$$= 2\tan\theta \Big|_0^{\pi/4} - 2\theta \Big|_0^{\pi/4} = \boxed{2 - \frac{\pi}{2}}$$

example

$$\int \frac{1}{(1-x^2)^{3/2}}\,dx$$

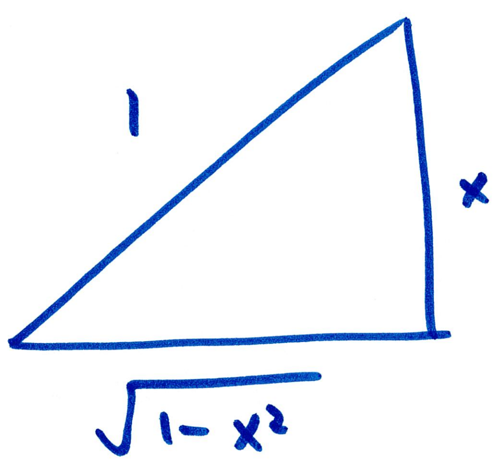
$$= \int \frac{1}{\left(\sqrt{1-x^2}\right)^3}\,dx$$

need triangle with sides

$\sqrt{1-x^2}$, x, 1

hypotenuse : 1, because $1^2 = x^2 + \left(\sqrt{1-x^2}\right)^2$

remaining sides : x, $\sqrt{1-x^2}$



we put the side that contains a constant as adjacent

(but it is not wrong to put x as adjacent)

Simplest way to relate x and θ: $\sin\theta = \frac{x}{1}$

$$x = \sin\theta$$
$$dx = \cos\theta \, d\theta$$

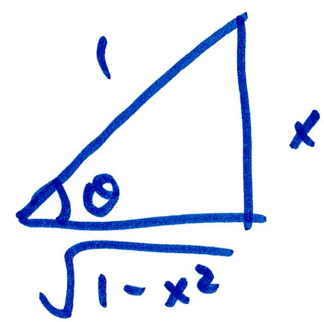
$$\int \frac{1}{(1-x^2)^{3/2}}\,dx = \int \frac{1}{(\underbrace{1-\sin^2\theta}_{\cos^2\theta})^{3/2}} \cdot \cos\theta \, d\theta$$

$$= \int \frac{1}{\cos^3\theta}\cos\theta \, d\theta = \int \frac{1}{\cos^2\theta}\,d\theta$$

$$= \int \sec^2\theta \, d\theta = \tan\theta + C$$

$\hookrightarrow \frac{\text{opp}}{\text{adj}}$

back to x



$$= \boxed{\frac{x}{\sqrt{1-x^2}} + C}$$