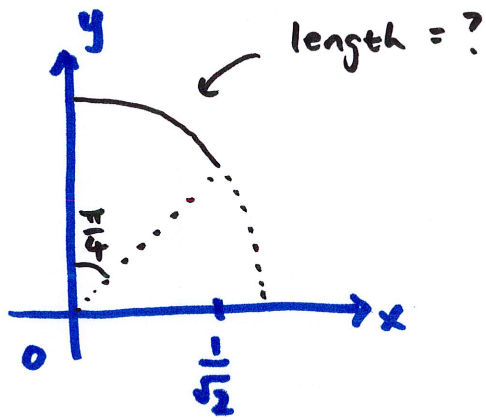
8.4 Trig Subs (part 2)

example Find the length of $y = \sqrt{1-x^2}$ from $x=0$ to $x = \frac{1}{\sqrt{2}}$

$$\int_a^b \sqrt{1+(y')^2}\,dx$$

$\underbrace{y=\sqrt{1-x^2}}$ part of circle
center $(0,0)$ radius 1
$x^2+y^2=1$



geometrically, we want is an eighth of the circumference of circle radius 1

answer: $2\pi \cdot (1)^2 \cdot \frac{1}{8} = \frac{\pi}{4}$

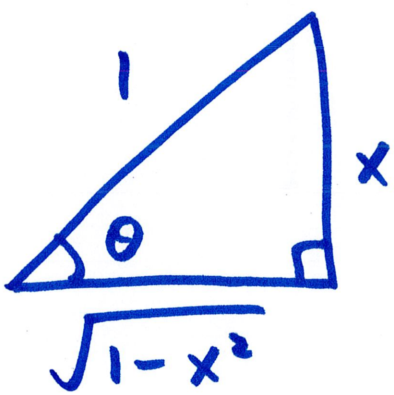
$$y = (1-x^2)^{1/2} \qquad y' = \frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{1-x^2}}$$

$$(y')^2 = \frac{x^2}{1-x^2} \qquad 1+(y')^2 = 1 + \frac{x^2}{1-x^2} = \frac{1}{1-x^2}$$

we integrate : $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \sqrt{\frac{1}{1-x^2}}\,dx = \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-x^2}}\,dx$ we couldn't do this before knowing trig subs

draw triangle w/ sides: $\sqrt{1-x^2}$, x , 1

hypotenuse : 1 because $1^2 = x^2 + (\sqrt{1-x^2})^2$



adjacent : $\sqrt{1-x^2}$

relate x and θ as simply as possible

$\sin\theta = \frac{x}{1}$

$x = \sin\theta \longrightarrow \theta = \sin^{-1}(x)$

$dx = \cos\theta\, d\theta$

bounds: $x = \frac{1}{\sqrt{2}} \rightarrow \theta = \sin^{-1}(\frac{1}{\sqrt{2}}) = \frac{\pi}{4}$

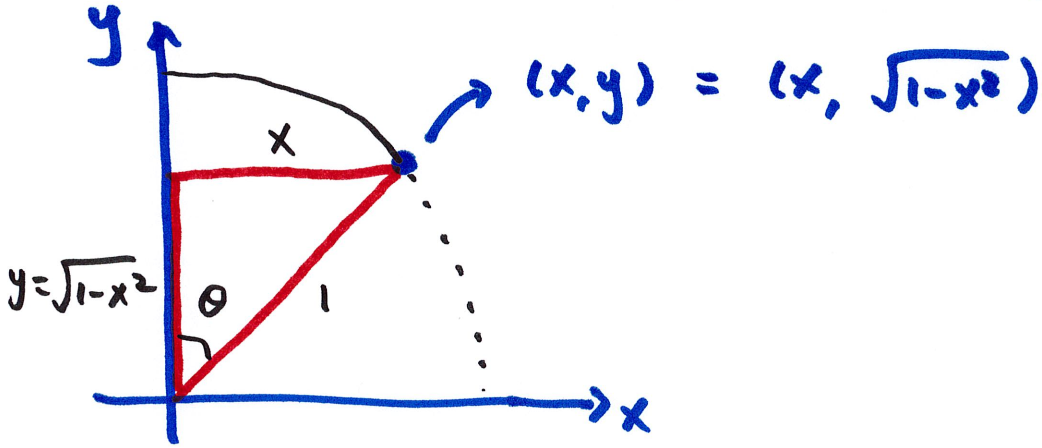
$x = 0 \rightarrow \theta = \sin^{-1}(0) = 0$

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{\underbrace{1-\sin^2\theta}_{\cos^2\theta}}} \cos\theta\, d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{\cos\theta} \cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} 1\, d\theta = \theta \Big|_0^{\frac{\pi}{4}} = \boxed{\frac{\pi}{4}}$$

the triangle is there in the geometric interpretation



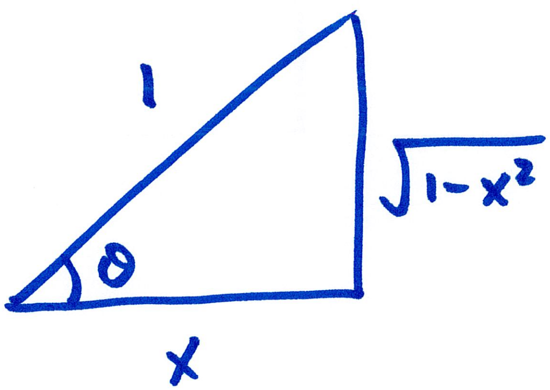
when doing trig subs, we have no flexibility with the hypotenuse, but we can place the other two however we want

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-x^2}}\,dx$$

triangle sides: $\sqrt{1-x^2}$, 1, x

hypotenuse: $\cancel{\sqrt{1-x^2}}$ 1

now try x as ~~hypotenuse~~ adjacent



interchanging ~~them~~ changes substitution

$$\cos\theta = \frac{x}{1}$$

$x = \cos\theta$

$dx = -\sin\theta\, d\theta$

bounds:

$x = \frac{1}{\sqrt{2}} \rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

$x = 0 \rightarrow \theta = \cos^{-1}(0) = \frac{\pi}{2}$

new integral

$$\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{1}{\sqrt{1-\cos^2\theta}} \cdot -\sin\theta \, d\theta = \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} -d\theta = -\theta \Big|_{\frac{\pi}{2}}^{\frac{\pi}{4}}$$

$$= -\left(\frac{\pi}{4} - \frac{\pi}{2}\right) = \boxed{\frac{\pi}{4}}$$

example

$$\int \frac{1}{x^2-6x+45}\,dx$$

$$= \int \frac{1}{\left(\sqrt{x^2-6x+45}\right)^2}\,dx$$

we want what's under the radical to be

sum/difference of squares

right now we don't have that

complete the square

$$x^2-6x+45 = x^2-6x+\left(\frac{-6}{2}\right)^2+45-\left(\frac{-6}{2}\right)^2$$

$$= x^2-6x+9+45-9$$

$$= x^2-6x+9+36 = (x-3)^2+6^2$$

$$\int \frac{1}{\left(\sqrt{(x-3)^2+6^2}\right)^2}\,dx$$

let $u = x - 3$

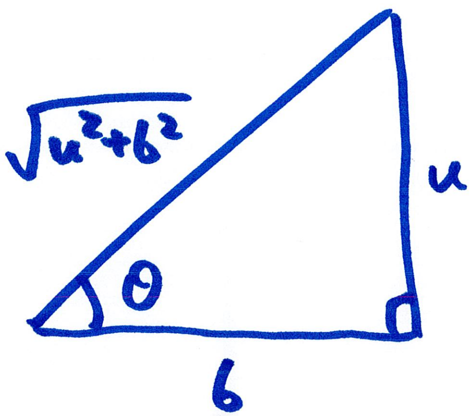
$du = dx$

$$\int \frac{1}{\left(\sqrt{u^2+6^2}\right)^2}\, du$$

triangle w/ sides

$\sqrt{u^2+6^2}$, u, 6

hypotenuse : $\sqrt{u^2+6^2}$



$\tan\theta = \dfrac{u}{6}$

$u = 6\tan\theta$

$du = 6\sec^2\theta\, d\theta$

$$\int \frac{1}{\left(\sqrt{36\tan^2\theta + 36}\right)^2}\, 6\sec^2\theta\, d\theta$$

$$\int \frac{1}{\left(\sqrt{36(\tan^2\theta+1)}\right)^2}\, 6\sec^2\theta\, d\theta$$

$$= \int \frac{1}{36\sec^2\theta}\, 6\sec^2\theta\, d\theta = \int \frac{1}{6}\, d\theta = \frac{1}{6}\theta + C$$

from $u = 6\tan\theta \rightarrow \theta = \tan^{-1}\left(\frac{u}{6}\right)$

$$= \frac{1}{6}\tan^{-1}\left(\frac{u}{6}\right) + C$$ then $u = x - 3$

$$= \boxed{\frac{1}{6}\tan^{-1}\left(\frac{x-3}{6}\right) + C}$$

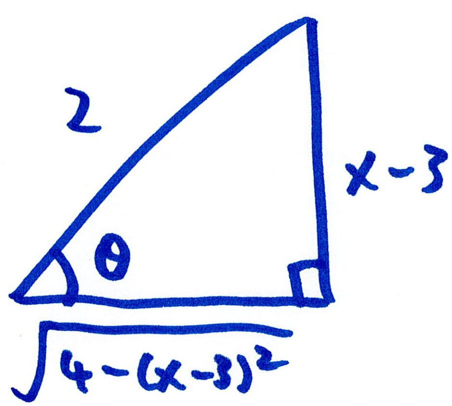
example

$$\int \frac{1}{\sqrt{(x-5)(1-x)}}\,dx$$

under radical needs to be sum/diff of squares

$$\begin{aligned}(x-5)(1-x) &= -x^2+6x-5\\ &= -(x^2-6x+5)\\ &= -(x^2-6x+9+5-9)\\ &= -\left[(x-3)^2-4\right] = 4-(x-3)^2\end{aligned}$$

triangle w/ sides $\sqrt{4-(x-3)^2}$, 2, $x-3$



$$\sin\theta = \frac{x-3}{2}$$

$$x-3 = 2\sin\theta$$

$$x = 2\sin\theta + 3$$

$$dx = 2\cos\theta\, d\theta$$

$$\int \frac{1}{\sqrt{4-(x-3)^2}}\,dx = \int \frac{1}{\sqrt{4-4\sin^2\theta}}\,2\cos\theta\,d\theta$$

$$= \int \frac{1}{2\cos\theta}\,2\cos\theta\,d\theta$$

$$= \int d\theta = \theta + C$$

$$\sin\theta = \frac{x-3}{2}$$

$$\theta = \sin^{-1}\left(\frac{x-3}{2}\right)$$

$$= \boxed{\sin^{-1}\left(\frac{x-3}{2}\right) + C}$$