8.5 Partial Fractions Expansion (part 1)

$$\int \frac{x-8}{x^2-7x+10}\,dx$$

$u = x^2-7x+10$

$du = (2x-7)\,dx$

we don't have what we need to form du easily

if we could recognize $\frac{x-8}{x^2-7x+10} = \frac{2}{x-2} - \frac{1}{x-5}$ → this is partial fractions expansion

then integration is easy

$$\int \frac{x-8}{x^2-7x+10}\,dx = \int \left(\frac{2}{x-2} - \frac{1}{x-5}\right) dx$$

$$= 2\int \frac{1}{x-2}\,dx - \int \frac{1}{x-5}\,dx$$

$u = x-2$ $\quad$ $u = x-5$

$\vdots$

$$= \boxed{2\ln|x-2| - \ln|x-5| + C}$$

How to do partial fraction expansions?

Case 1: Denominator is a product of <u>distinct</u> <u>linear</u> factors

not repeating   power is 1

$$\frac{x-8}{x^2-7x+10} = \frac{x-8}{(x-5)(x-2)} = \frac{A}{x-5} + \frac{B}{x-2}$$

A, B are constants

(just like $\frac{1}{6} = \frac{A}{3} + \frac{B}{2}$)

now find A, B

$$\frac{x-8}{(x-5)(x-2)} = \frac{A}{x-5} + \frac{B}{x-2}$$

multiply by $(x-5)(x-2)$

$$x-8 = A(x-2) + B(x-5)$$

$$= Ax - 2A + Bx - 5B$$

$$x-8 = (A+B)x + (-2A-5B)$$

true only if coefficients match

$A+B = 1$ — ①

$-2A-5B = -8$ — ②

from ① $B = 1 - A$ sub into ②

$$-2A - 5(1-A) = -8$$

$$-2A - 5 + 5A = -8$$

$3A = -3$ $A = -1$ then $B = 1 - A = 1 - (-1) = 2$

so, $$\frac{x-8}{x^2-7x+10} = \frac{-1}{x-5} + \frac{2}{x-2}$$

example $$\int \frac{1}{x^2-4}\,dx$$

$$\frac{1}{x^2-4} = \frac{1}{\underbrace{(x-2)(x+2)}_{\text{distinct, linear}}} = \frac{A}{x-2} + \frac{B}{x+2}$$

multiply by $(x-2)(x+2)$

$$1 = (x-2)(x+2)\left[\frac{A}{x-2} + \frac{B}{x+2}\right]$$

$$1 = A(x+2) + B(x-2)$$

$$0x + 1 = (A+B)x + (2A - 2B)$$

$A + B = 0 \quad - ①$

$2A - 2B = 1 \quad - ②$

multiply ① by 2

$2A + 2B = 0 \quad - ③$

add ②, ③

$4A = 1 \rightarrow A = 1/4$ then from ① $B = -1/4$

So,
$$\int \frac{1}{x^2-4}\,dx = \int \left(\frac{1/4}{x-2} + \frac{-1/4}{x+2}\right) dx$$

$$= \frac{1}{4}\int \frac{1}{x-2}\,dx - \frac{1}{4}\int \frac{1}{x+2}\,dx$$

$$= \boxed{\frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C}$$

Case 2: Denominator is a product of linear factors and some of which are repeated.

$$\frac{10}{5x^2-2x^3} = \frac{10}{(x^2)(5-2x)} = \frac{10}{\underbrace{(x)(x)}_{\text{repeated linear factor}}(5-2x)}$$

$$\frac{10}{(x)(x)(5-2x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{5-2x}$$

↳ x is repeated so this additional one is needed

multiply by $(x)(x)(5-2x)$

$$10 = A(x)(5-2x) + B(5-2x) + C(x^2)$$

here is another way to solve for these constants

choose a number for x so that at least one term on the right disappears choose $x = 0$

$$10 = B(5-0) \rightarrow B = 2$$

choose $x = 5/2$

$$10 = c\,(5/2)^2 = \frac{25}{4}c \rightarrow c = \frac{8}{5}$$

now I know two out of three constants, choose another convenient x to solve for the remaining one

choose $x = 1$

$$10 = A(1)(5-2) + \overset{B}{(2)}(5-2) + \overset{C}{\left(\frac{8}{5}\right)}(1)^2 \rightarrow A = \frac{4}{5}$$

so,
$$\int \frac{10}{5x^2 - 2x^3}\,dx = \int\left(\frac{4/5}{x} + \frac{2}{x^2} + \frac{8/5}{5-2x}\right)dx$$

$$= \frac{4}{5}\int \frac{1}{x}\,dx + 2\int \frac{1}{x^2}\,dx + \frac{8}{5}\int \frac{1}{5-2x}\,dx$$

$$= \boxed{\frac{4}{5}\ln|x| + 2\cdot\left(-\frac{1}{x}\right) + \frac{8}{5}\cdot -\frac{1}{2}\ln|5-2x| + C}$$

important : BEFORE doing the expansion make sure the numerator has a lower degree than the denominator

$$\frac{x-8}{x^2-7x+10}$$ is ok.

first degree

second degree

$$\frac{-2x^3+5x^2+10}{-2x^3+5x^2}$$

3rd

3rd

numerator does NOT have lower degree

Cannot expand right away

$$\frac{-2x^3+5x^2}{-2x^3+5x^2}+\frac{10}{-2x^3+5x^2} = 1+\underbrace{\frac{10}{-2x^3+5x^2}}_{\text{can expand this}}$$

0th

3rd

example

$$\frac{x^2+1}{x^2+x+3}$$

2nd

2nd

numerator MUST have lower degree

$$= \frac{(x^2+1+x+2)-x-2}{x^2+x+3}$$

$$= \frac{x^2+1+x+2}{x^2+x+3} + \frac{-x-2}{x^2+x+3}$$

$$= 1 + \frac{-x-2}{x^2+x+3}$$

1st

2nd