8.5 Partial Fractions (part 2)

remember, to do an expansion, the numerator MUST have lower degree

$$\frac{26x^3-52x^2+2}{x^2-2x}$$

← degree 3

← degree 2

long division is another way to reduce

x^2-2x goes into $26x^3-52x^2+2$ $26x$ times

$$\begin{array}{r l} & \quad 26x \\ x^2-2x & \overline{\big)\, 26x^3-52x^2+2} \\ & -(26x^3-52x^2 \quad\;\;) \\ \hline & \qquad\qquad\qquad\quad 2 \end{array}$$

← remainder

$$= 26x + \frac{2}{x^2-2x}$$

as a quick review, let's do $\displaystyle\int \frac{26x^3-52x^2+2}{x^2-2x}\,dx$

$$= \int \left(26x + \frac{2}{x^2-2x}\right)dx = \int 26x\,dx + \int \frac{2}{x(x-2)}\,dx$$

$$= \int 26x\,dx + \int \left(\frac{1}{x-2} - \frac{1}{x}\right)dx$$

$$= \boxed{13x^2 + \ln|x-2| - \ln|x| + C}$$

now we look at cases with quadratic factors in denominator

reducible → turns into linear factors, we know how to handle

irreducible → cannot be factored

$$\frac{2x^2-x+4}{x^3+4x} = \frac{2x^2-x+4}{\underbrace{(x)}_{\text{linear}}\underbrace{(x^2+4)}_{\text{irreducible}}}$$

$$= \frac{A}{\underbrace{x}_{\substack{\text{for linear} \\ \text{factor } x \\ \text{in denominator}}}} + \frac{Bx+C}{x^2+4}$$

linear form
in numerator
of irreducible quadratic

example

$$\int \frac{x^2+x+2}{(x+1)(x^2+1)}\,dx$$

degree check : numerator is 2

denominator is 3

ok.

$$\frac{x^2+x+2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

find A, B, C

$$x^2+x+2 = (x+1)(x^2+1)\left(\frac{A}{x+1} + \frac{Bx+C}{x^2+1}\right)$$

$$= A(x^2+1) + (Bx+C)(x+1)$$

$$= A(x^2+1) + Bx^2 + Bx + Cx + C$$

$$x^2+x+2 = (A+B)x^2 + (B+C)x + (A+C)$$

one way: solve system

$$A+B = 1$$

$$B+C = 1$$

$$A+C = 2$$

another way: choose x and solve

let's do the choose x way

$x=0$: $2 = A+C$ $\quad \boxed{C=1}$

$x=-1$: $2 = 2A \rightarrow \boxed{A=1}$

$x=1$: $4 = (1)(2) + (B+1)(2)$

$= 2 + 2B + 2$

$\boxed{B=0}$

$$\int \frac{x^2+x+2}{(x+1)(x^2+1)}\,dx = \int \left(\frac{1}{x+1} + \frac{0x+1}{x^2+1}\right)dx$$

$$= \int \frac{1}{x+1}\,dx + \int \underbrace{\frac{1}{x^2+1}}\,dx = \boxed{\ln|x+1| + \tan^{-1}(x) + D}$$

either recognize as $\tan^{-1}(x)$

or write as $\int \frac{1}{(\sqrt{x^2+1})^2}\,dx$ and do a trig sub.

repeated irreducible quadratic factors are handled the same way as repeated linear factors

$$\frac{1}{(x)(x^2+1)^2} = \frac{1}{(x)(x^2+1)(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$1 = (x)(x^2+1)^2 \left(\frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} \right)$$

$$= A(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)(x)$$

$$1 = (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

$A+B=0 \rightarrow$ knowing $A=1$, $\boxed{B=-1}$

$\boxed{C=0}$

$2A+B+D=0 \rightarrow$ knowing A, B, $\boxed{D=-1}$

$C+E=0 \rightarrow$ knowing $C=0$, $\boxed{E=0}$

$\boxed{A=1}$

so, $$\int \frac{1}{x(x^2+1)^2}\,dx$$

$$= \int \left(\frac{1}{x} + \frac{-1\cdot x + 0}{x^2+1} + \frac{-1\cdot x+0}{(x^2+1)^2} \right) dx$$

$$= \int \frac{1}{x}\,dx \underbrace{- \int \frac{x}{x^2+1}\,dx - \int \frac{x}{(x^2+1)^2}\,dx}_{\substack{u = x^2+1 \\ du = 2x\,dx \\ \vdots \\ \text{etc}}}$$

$$= \boxed{\ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2}\,\frac{1}{x^2+1} + K}$$

write down the FORM of the expansion:

$$\frac{2x^3+5x-10}{x^2(x-3)(x^2+1)^3} \to \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2}$$

$$+ \frac{Hx+I}{(x^2+1)^3}$$