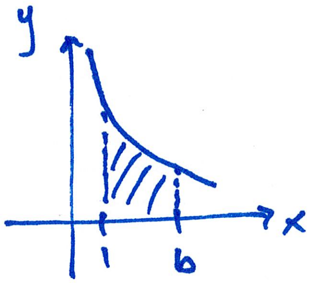
8.9 Improper Integrals

$f(x) = \frac{1}{x}$

we know $\int_1^b \frac{1}{x}\,dx$ → area under $f(x) = \frac{1}{x}$ from $x=1$ to $x=b$



$$\int_1^b \frac{1}{x}\,dx = \ln|x|\Big|_1^b = \ln b - \cancel{\ln 1}^{\,0} = \ln b$$

what happens if $b \to \infty$?

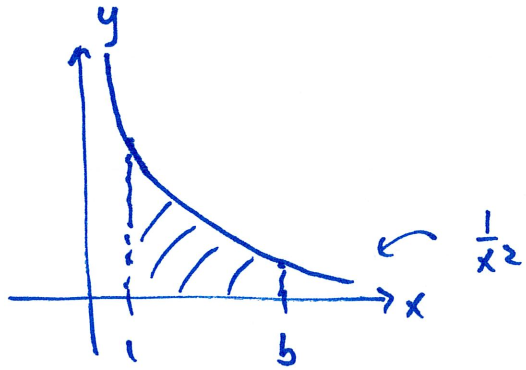
area is $\ln b$ so as $b \to \infty$, $\ln b \to \infty$ this means the area under $f(x) = \frac{1}{x}$ from 1 to b is unbounded as $b \to \infty$

$\lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$ gives us this unbounded area

we usually write as $\int_1^\infty \frac{1}{x}\,dx$ this is one type of Improper Integral : when one of (or both) the bounds is ∞ or $-\infty$

another example : $f(x) = \frac{1}{x^2}$ from $x=1$ to $x=b$

$$\int_1^b \frac{1}{x^2}\,dx$$



$$\int_1^b x^{-2}\,dx = -x^{-1}\Big|_1^b = -\frac{1}{x}\Big|_1^b$$

$$= -\frac{1}{b} + 1 = 1 - \frac{1}{b}$$

now notice as $b \to \infty$

$$\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \int_1^\infty \frac{1}{x^2}\,dx = 1$$

as $b \to \infty$, the area under $\frac{1}{x^2}$ gets closer and closer to 1

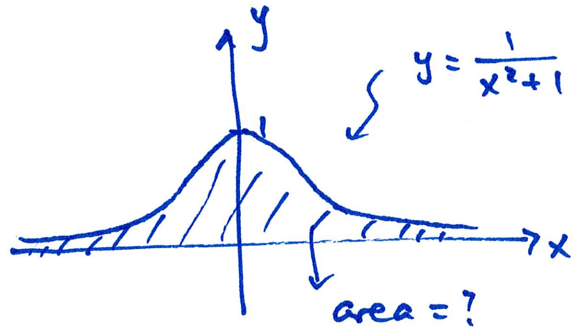
if the improper integral gives ∞ or $-\infty$, we say the integral <u>diverges</u> (or is <u>divergent</u>) e.g. $\int_1^\infty \frac{1}{x}\,dx$

if the improper integral gives a number, we say the integral <u>converges</u> (or is <u>convergent</u>) e.g. $\int_1^\infty \frac{1}{x^2}\,dx$

it turns out $\displaystyle\int_a^\infty \frac{1}{x^p}\,dx$ converges if $p>1$, diverges if $p\le 1$ — determined by how fast $\frac{1}{x^p}$ decreases

if both bounds are $\pm\infty$, we handle the integral in a similar way

<u>example</u> $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$



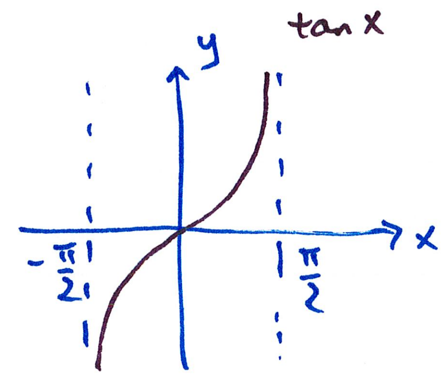
$$= \int_{-\infty}^{0} \frac{1}{x^2+1}\,dx + \int_0^\infty \frac{1}{x^2+1}\,dx$$

we can't directly work with ∞ or $-\infty$, so introduce "a" and "b"

$$= \lim_{a\to-\infty} \int_a^0 \frac{1}{x^2+1}\,dx + \lim_{b\to\infty} \int_0^b \frac{1}{x^2+1}\,dx$$

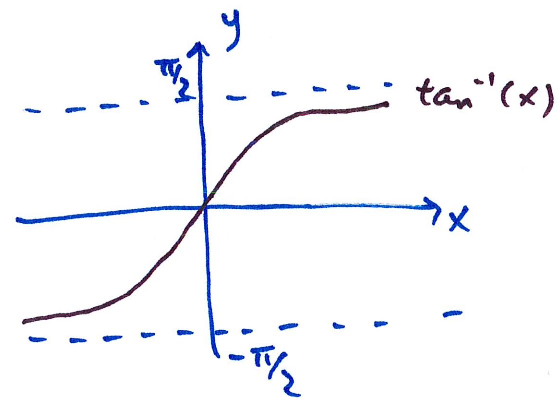
$$= \lim_{a\to\infty} \tan^{-1}(x)\Big|_a^0 + \lim_{b\to\infty} \tan^{-1}(x)\Big|_0^b$$

$$= \lim_{a\to-\infty}\left(\cancelto{0}{\tan^{-1}(0)} - \tan^{-1}(a)\right) + \lim_{b\to\infty}\left(\tan^{-1}(b) - \cancelto{0}{\tan^{-1}(0)}\right)$$



$$= \lim_{a\to-\infty}\left(-\tan^{-1}(a)\right) + \lim_{b\to\infty}\tan^{-1}(b)$$

$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \boxed{\pi}$$



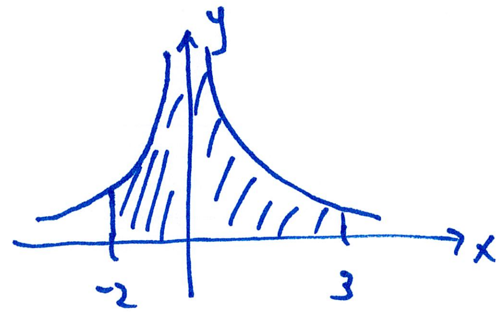
So,
$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \pi$$

this integral is convergent

the second kind of improper integral is where the integrand becomes ~~unb~~ unbounded at some point in $\int_a^b f(x)\,dx$

(so $f(x) \to \infty$ or $-\infty$ somewhere on $a < x < b$)

<u>example</u> $\int_{-2}^{3} \frac{1}{x^4}\,dx$



$\frac{1}{x^4} \to \infty$ as $x \to 0$ which is in $-2 < x < 3$

we cannot integrate <u>through</u> 0

we want to "avoid" 0

$\int_{-2}^{0} \frac{1}{x^4}\,dx + \int_{0}^{3} \frac{1}{x^4}\,dx$ but 0 is still bad

$$= \lim_{b \to 0^-} \int_{-2}^{b} \frac{1}{x^4}\,dx + \lim_{a \to 0^+} \int_{a}^{3} \frac{1}{x^4}\,dx$$

$$= \lim_{b\to 0^-}\left(-\frac{1}{3x^3}\right)\Big|_{-2}^{b} + \lim_{a\to 0^+}\left(-\frac{1}{3x^3}\right)\Big|_{a}^{3}$$

$$= \underbrace{\lim_{b\to 0^-}}_{\substack{\text{b is a}\\ \text{small}\\ \text{negative \#}}}\left(-\underbrace{\frac{1}{3b^3}}_{\substack{\text{large}\\ \text{positive \#}\\ (\infty)}} - \frac{1}{24}\right) + \underbrace{\lim_{a\to 0^+}}_{\substack{\text{small}\\ \text{pos. \#}}}\left(-\frac{1}{81} + \underbrace{\frac{1}{3a^3}}_{\substack{\text{large pos. \#}\\ (\infty)}}\right)$$

$$= \infty - \frac{1}{24} + -\frac{1}{81} + \infty = \boxed{\infty}$$ so integral diverges

what if we integrated through 0 w/o realizing $\frac{1}{x^4}$ has a problem?

$$\int_{-2}^{3} \frac{1}{x^4}\,dx = -\frac{1}{3x^3}\Big|_{-2}^{3} = -\frac{1}{81} - \frac{1}{24} = -\frac{35}{648}$$ wrong!

we must handle improper integrals properly!

Sometimes we can compare improper integrals to know if they converge or diverge.

For example, from earlier, we found $\int_1^\infty \frac{1}{x^2}\,dx = 1$ converges

since $\frac{1}{x^2+1} \le \frac{1}{x^2}$ because $\frac{1}{x^2+1}$ has bigger denominator

so $\int_1^\infty \frac{1}{x^2+1}\,dx \le \int_1^\infty \frac{1}{x^2}\,dx$

this allows me to determine that $\int_1^\infty \frac{1}{x^2+1}\,dx$ converges w/o integrating

similarly, we know $\int_1^\infty \frac{1}{x}\,dx$ diverges

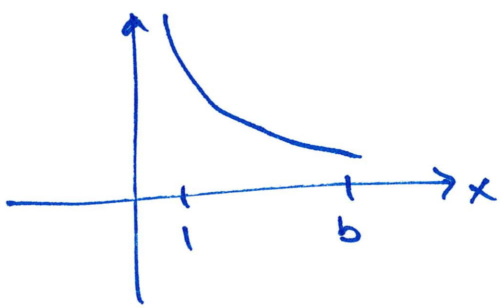
since $\frac{1}{x-\frac{1}{2}} \ge \frac{1}{x}$ since $\frac{1}{x-\frac{1}{2}}$ has smaller denominator

$$\int_1^\infty \frac{1}{x-\frac{1}{2}}\,dx \ge \int_1^\infty \frac{1}{x}\,dx$$

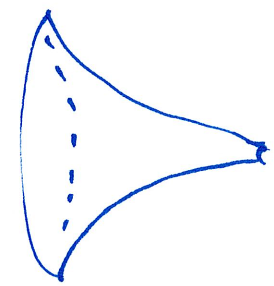
so again, w/o any integration, we know $\int_1^\infty \frac{1}{x-\frac{1}{2}}\,dx$ diverges

Gabriel's Horn

$f(x) = \frac{1}{x}$



revolve around x-axis



as $b \to \infty$, the solid will have a <u>finite volume</u> but <u>infinite surface area</u>