10.1 Sequences and Series: An Overview

a **sequence** is a list of numbers in a particular order

for example, $\{1, 2, 3, 4, 5, 6, \ldots\}$ natural numbers

$\{2, 4, 6, 8, 10, \ldots\}$ even numbers

we can list the terms like above or in explicit formula

for example, $\{1, 2, 3, 4, 5, \ldots\} = \{a_n\}_{n=1}^{\infty}$

end

start at $n=1$

a_1 a_2

a is the name of this sequence

a_n is the n^{th} term

$= \{n\}_{n=1}^{\infty}$

$\{2, 4, 6, 8, 10, \ldots\} = \{2n\}_{n=1}^{\infty}$

Sometimes we use the recurrence relation to list terms

e.g. $\{1, 2, 3, 4, 5, \dots\}$

can be expressed as $a_1 = 1$, $a_{n+1} = a_n + 1$

↳ for example, $a_3 = a_2 + 1$

3rd term is 2nd term + 1

e.g. $\{2, 4, 6, 8, \dots\}$

$a_1 = 2$, $a_{n+1} = a_n + 2$

$\{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$ Fibonacci Sequence

each is sum of previous two

$a_1 = 1$, $a_2 = 1$, $a_{n+2} = a_{n+1} + a_n$ $n = 1, 2, 3, \dots$

if we want a_4

then $a_4 = a_3 + a_2$ with $n = 2$

$= 3$

a sequence is said to <u>converge</u> (or is convergent)

if $\lim_{n\to\infty} a_n$ exists. This limit is called the limit of the sequence

if $\lim_{n\to\infty} a_n$ DNE, the sequence is <u>divergent</u> (or diverges)

<u>example</u> $a_n = \dfrac{(-1)^n n}{2n^2+1}$ $\quad n = 1, 2, 3, 4, \ldots$

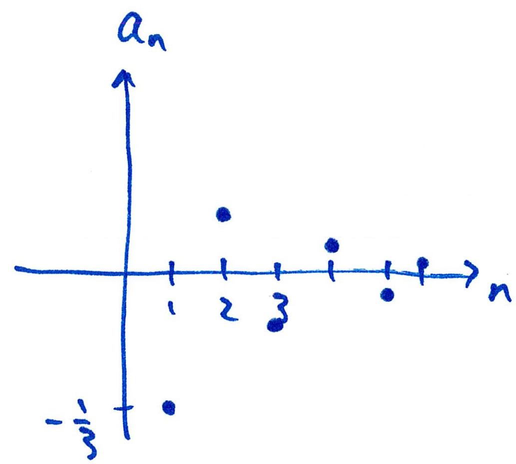
list a few: $a_1 = \dfrac{(-1)^1 \cdot 1}{2n^1+1} = -\dfrac{1}{3}$

$a_2 = \dfrac{2}{9}$

$a_3 = \dfrac{-3}{19}$

$a_4 = \dfrac{4}{33}$

$a_5 = \dfrac{-5}{51}$



the magnitude ($|a_n|$) seems to decrease as n increases

$$a_n = \frac{(-1)^n n}{2n^2+1}$$

$$\lim_{n\to\infty} \frac{(-1)^n n}{2n^2+1} = \underbrace{\lim_{n\to\infty} (-1)^n}_{\substack{\text{flips sign} \\ \text{does not} \\ \text{change magnitude}}} \cdot \underbrace{\lim_{n\to\infty} \frac{n}{2n^2+1}}_{\text{goes to } 0} = 0$$

Since $\lim_{n\to\infty} a_n = 0$ (exists) we say the sequence $a_n = \frac{(-1)^n n}{2n^2+1}$ <u>converges</u>

and limit of sequence is 0.

How about $a_n = \frac{n}{n+1}$ $n = 1, 2, 3, \ldots$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1}$$

let's review indeterminate forms and limits

if limit appears to go to $\frac{\infty}{\infty}$ or $\frac{0}{0}$, it is indeterminate

we can use l'Hospital's Rule to find limit

$$\lim_{n\to\infty} \frac{f(x)}{g(x)} = \lim_{n\to\infty} \frac{f'(x)}{g'(x)}$$

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1} = 1$$

or, we can reason it this way: $\frac{n}{n+1}$

when n is a large number

$$\frac{10{,}000}{10{,}001}$$

$$\frac{100{,}000}{100{,}001}$$

numerator $\approx$ denominator when n is large

so limit is 1

a **series** is the sum of the numbers in a sequence

for example, $1+2+3+4+5+\dots$ terms from $\{1, 2, 3, 4, \dots\}$ added up

like with sequences, we can write series compactly

end

Sigma : means to add

$$1+2+3+4+5+\dots = \sum_{n=1}^{\infty} n$$

what each term looks like

n=1 n=2 n=3

start at n=1

if the terms don't end, it's an **infinite** series

if they end, it's a **finite** series

e.g. $$\sum_{n=1}^{5} n = 1+2+3+4+5$$

n=3

n=1 n=2

example

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$= \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \cdots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots$$

$n=1$ $\quad n=2$ $\quad n=3$

the sum of the first n terms is called the n^{th} partial sum, S_n

first partial sum

sum of first 1 term

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$

$$\vdots$$

$$S_{10} = \cdots = \frac{1023}{1024}$$

note the trend in S_n: S_n appears to go to 1 as n increases

numerator $= 2^n - 1$

denominator $= 2^n$

so the n^{th} partial sum is $S_n = \dfrac{2^n - 1}{2^n}$

notice $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \dfrac{2^n - 1}{2^n} = 1$

the partial sums get closer to 1

the sum of the infinite series is approaching but never exceeding or equaling 1

if $\lim_{n\to\infty} S_n$ exists, we say the series converges (or is convergent)

if $\lim_{n\to\infty} S_n$ DNE, the series diverges (or is divergent)

is $\sum_{n=1}^{\infty} n$ convergent? no, because $1+2+3+4+\cdots \to \infty$

example $\sum_{n=1}^{\infty} \cos(n\pi)$

$= -1 + 1 - 1 + 1 - 1 + 1 - \cdots = \sum_{n=1}^{\infty} (-1)^n$

$n=1$ $n=2$

partial sums

$S_1 = -1$

$S_2 = 0$

$S_3 = -1$

$S_4 = 0$

$S_5 = -1$

$\vdots$

does the series converge?

NO, since S_n does not have a limit

alternatively, it does not converge because the sequence of the partial sum does not converge

$\{S_1, S_2, S_3, \dots\}$

$= \{-1, 0, -1, 0, -1, \dots\}$ has no limit