10.2 Sequences

sequence : list of numbers in some particular order

$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \ldots\}$$

sequence converges if $\lim\limits_{n\to\infty} a_n$ exists

for example, $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{99{,}999}{100{,}000}, \ldots\right\}$

the numbers approach 1 $\rightarrow$ ~~th~~ sequence converges to 1

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

<u>example</u> $\left\{ n^{1/n} \right\}_{n=1}^{\infty}$

$= \left\{ 1, 2^{1/2}, 3^{1/3}, 4^{1/4}, \dots \right\}$

initially gets bigger, but then decrease

but is there a limit ?

$\lim_{n \to \infty} n^{1/n} \to \infty^0 = ?$

∞^0 indeterminate form

we can use l'Hospital's Rule

$\frac{0}{0}$ or $\frac{\infty}{\infty}$

but ∞^0 needs to be turned into $\frac{0}{0}$ or $\frac{\infty}{\infty}$ first

$y = n^{1/n}$

then $\ln y = \ln n^{1/n} = \frac{1}{n} \ln n = \frac{\ln(n)}{n} \to \frac{\infty}{\infty}$ as $n \to \infty$

now we find $\lim_{n \to \infty} \ln y$ using l'Hospital's Rule

$$\lim_{n\to\infty} \frac{\ln(n)}{n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{1}{n} = 0$$

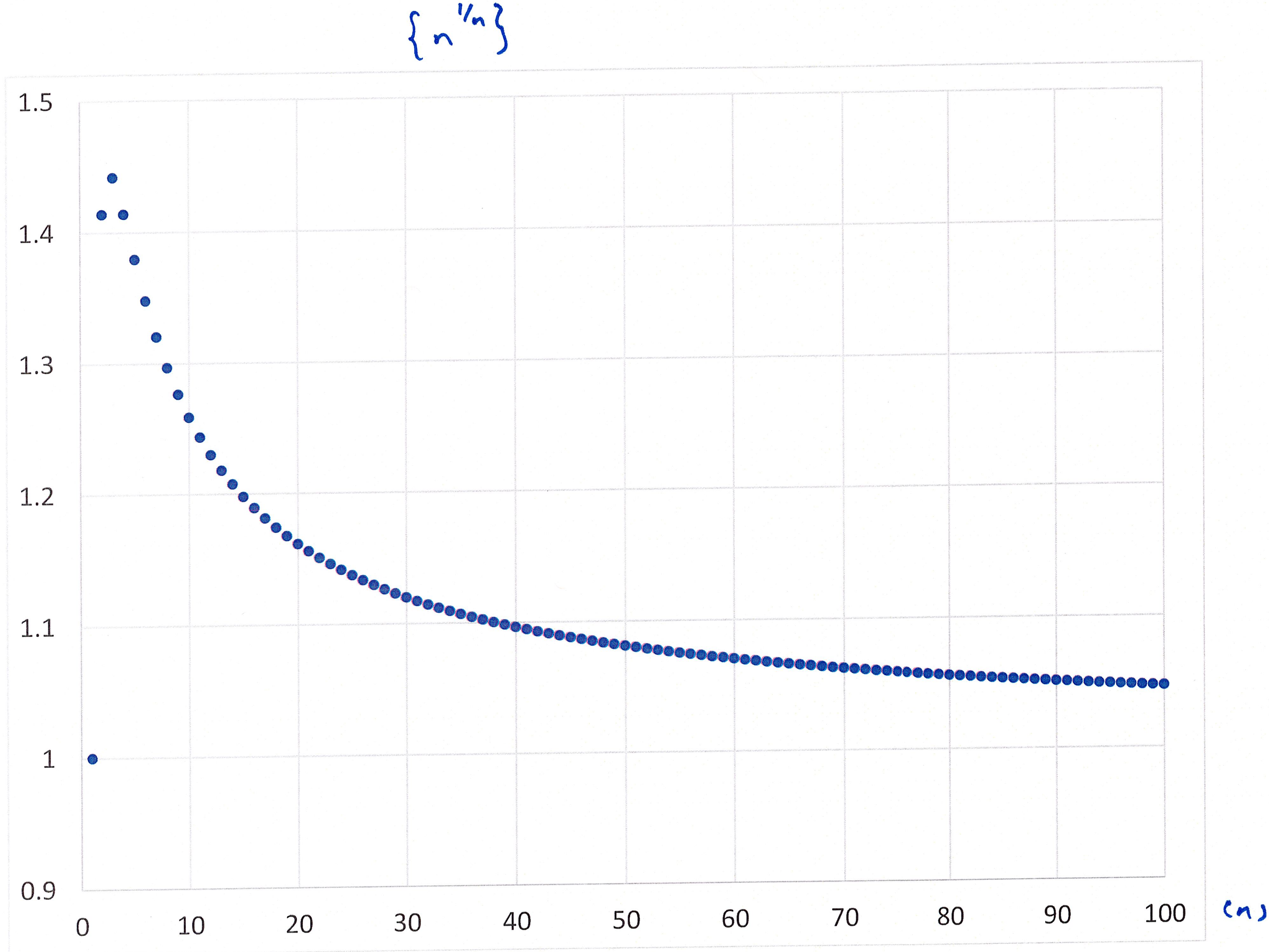
now undo the ln

we found $\lim_{n\to\infty} \ln y = 0$ but we want $\lim_{n\to\infty} y = \lim_{n\to\infty} n^{1/n}$

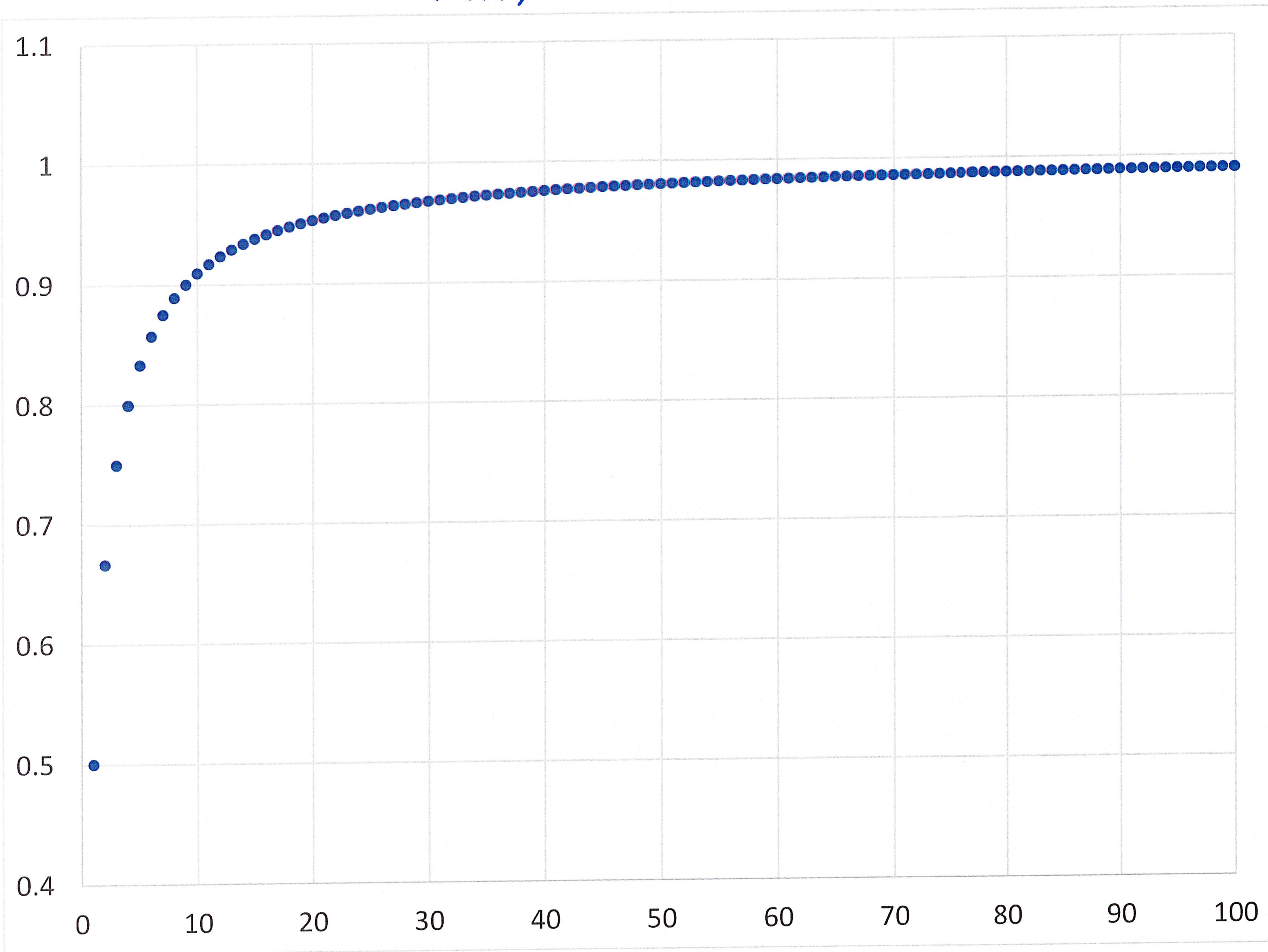
since $e^{\ln y} = y$ we can find $\lim_{n\to\infty} y$ from $\lim_{n\to\infty} \ln y$

$$\lim_{n\to\infty} e^{\ln y} = e^0 \to \lim_{n\to\infty} y = 1$$

so, $\{1, 2^{1/2}, 3^{1/3}, 4^{1/4}, \ldots\}$ has limit of 1 (converges)


$\{n^{1/n}\}$
1.5
1.4
1.3
1.2
1.1
1
0.9
0
10
20
30
40
50
60
70
80
90
100
(n)

$\left\{ \frac{n}{n+1} \right\}$



notice $\{ n^{1/n} \}$ increases initially then decreases

whereas $\left\{ \frac{n}{n+1} \right\}$ always increases

if a sequence increases/decreases for <u>all n</u>, we say the sequence is <u>monotonic</u>

$\{1, 2, 3, 4, 5, \ldots\}$ is monotonic

$\{-1, -2, -3, -4, -5, \ldots\}$ is monotonic

$\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, \ldots\}$ is not monotonic

$$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$$

$$\left\{n^{1/n}\right\}_{n=1}^{\infty} = \left\{1, 2^{1/2}, 3^{1/3}, 4^{1/4}, \ldots\right\}$$

are both bounded because all terms are no bigger or smaller than some number

$\left\{\frac{n}{n+1}\right\}$ terms are no bigger than 1 and no smaller than $1/2$

$\left\{n^{1/n}\right\}$ terms are no smaller than 1 and no bigger than $3^{1/3}$

and it turns out that if a sequence is bounded and monotonic, then it converges an important sequence of the form $\{r^n\}$ is called geometric sequence

→ all terms have a common ratio r

$$\left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots \right\} = \left\{ \left(\frac{1}{2}\right)^n \right\}_{n=1}^{\infty}$$

mult. by $1/2$ mult. by $1/2$

$$\left\{ \left(-\frac{1}{3}\right)^n \right\}_{n=1}^{\infty} = \left\{ -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \dots \right\}$$

a geometric sequence converges if $-1 < r \leq 1$ (1 included by not -1)

$r = 1$: $\{1, 1, 1, 1, 1, \dots\}$ limit is 1 so converges

$r = -1$: $\{-1, 1, -1, 1, -1, 1, \dots\}$ no limit so diverges

as long as $|r| < 1$, $\{r^n\}$ converges since $\lim\limits_{n \to \infty} r^n = 0$

a lot of times we see factorial (!) in sequences and series

<u>example</u> $\left\{ \frac{n}{n!} \right\}_{n=1}^{\infty}$ converges?

factorial: $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot \cdots 1$

$n! = (n)(n-1)(n-2)(n-3)\cdots(1)$

$= \left\{ 1, \frac{2}{2!}, \frac{3}{3!}, \frac{4}{4!}, \ldots \right\}$

$= \left\{ 1, 1, \frac{1}{2}, \frac{1}{6}, \ldots \right\}$

the terms seem to decrease, but to what? and does it alway decrease?

we need to see $\lim\limits_{n\to\infty} \frac{n}{n!} = L$ ← some number

$$\lim_{n\to\infty} \frac{n}{n!} = \lim_{n\to\infty} \frac{(\cancel{n})}{(\cancel{n})(n-1)(n-2)(n-3)\cdots(1)} = \lim_{n\to\infty} \frac{1}{(n-1)(n-2)\cdots(1)} = 0$$

so now we know for sure the terms approach 0

the sequence converges

<u>example</u> $\left\{ \frac{e^n}{n!} \right\}_{n=1}^{\infty}$

$$= \left\{ \frac{e^1}{1!}, \frac{e^2}{2!}, \frac{e^3}{3!}, \cdots \right\}$$

numerator appears to dominate initially

how about for bigger n?

$n=10$: $e^{10} = 22{,}026$

$10! = 3{,}628{,}800$

$n=100$: $e^{100} \approx 3 \times 10^{43}$

$100! = 9 \times 10^{157}$

appears the denominator eventually dominates

but can we be sure?

does $\lim_{n\to\infty} \frac{e^n}{n!}$ exist?

$$\frac{e^n}{n!} = \frac{e^n}{(n)(n-1)(n-2)(n-3)\cdots(1)}$$

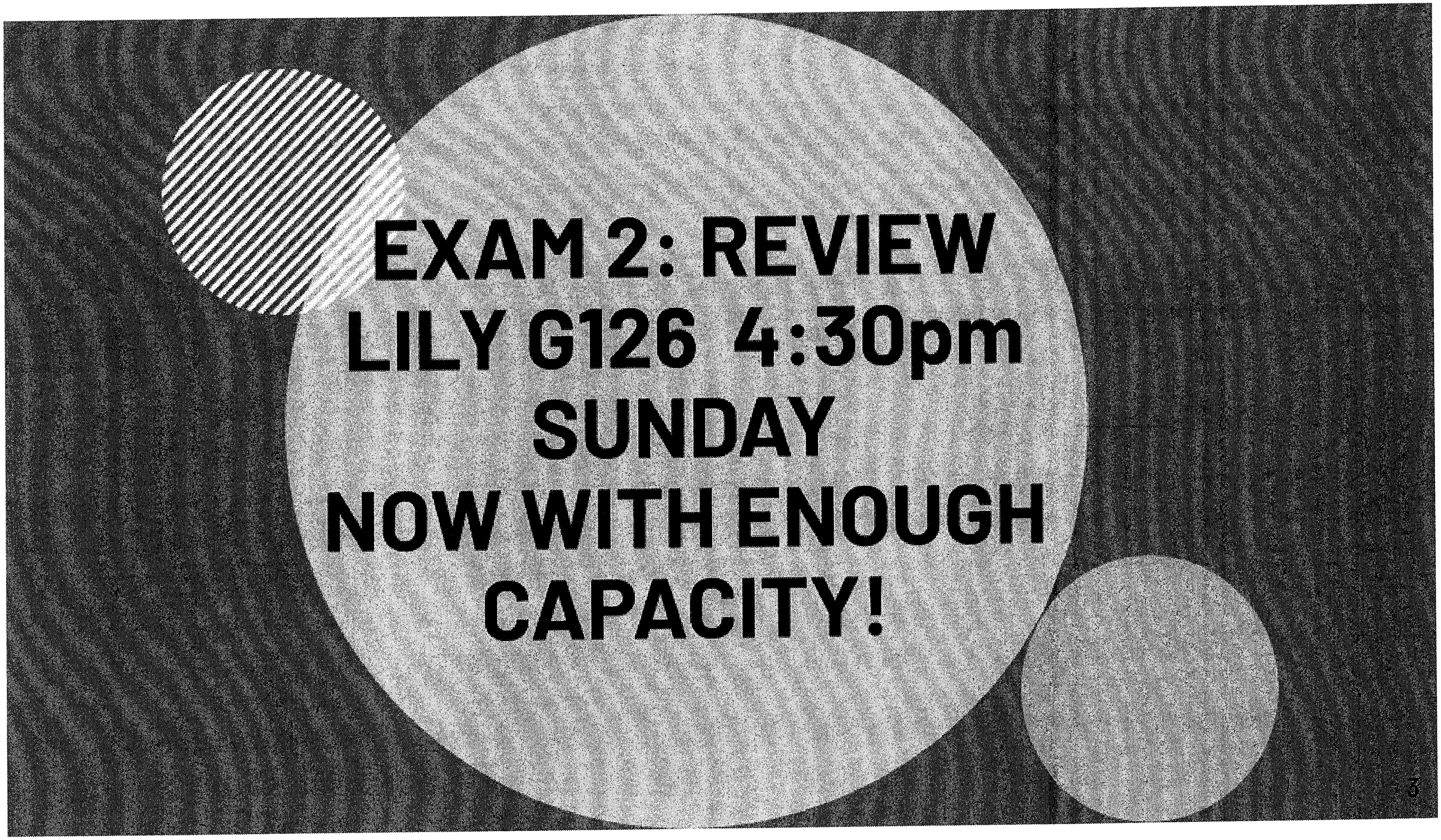
when n is large, the denominator is a polynomial with leading term n^n

$$\frac{e^n}{n!} \approx \frac{e^n}{n^n} = \left(\frac{e}{n}\right)^n \rightarrow \text{looks like } r^n$$

when $n \to \infty$, $\frac{e}{n} \to 0$

so terms of this sequence eventually act like terms in a geometric sequence r^n with $|r| < 1$

so this sequence converges


EXAM 2: REVIEW
LILY G126 4:30pm
SUNDAY
NOW WITH ENOUGH
CAPACITY!
3