10.3 Infinite Series

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \cdots$$

today : geometric series
telescoping series

geometric: fixed ratio between succeeding terms

for example, $\frac{1}{4} + \frac{1}{12} + \frac{1}{36} + \frac{1}{108} + \cdots$ ratio $= r = \frac{1}{3}$

we can write a geometric series as

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{k=0}^{\infty} ar^k$$

$$\frac{1}{4} + \frac{1}{12} + \frac{1}{36} + \frac{1}{108} + \cdots = \frac{1}{4}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots\right)$$

$$= \frac{1}{4} + \frac{1}{4}\left(\frac{1}{3}\right) + \frac{1}{4}\left(\frac{1}{3}\right)^2 + \frac{1}{4}\left(\frac{1}{3}\right)^3 + \cdots$$

$a = \frac{1}{4}$, $r = \frac{1}{3}$

$$= \sum_{k=0}^{\infty} \frac{1}{4}\left(\frac{1}{3}\right)^k$$

↖ starting index

the starting index is not important

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots$$

notice $\displaystyle\sum_{k=1}^{\infty} ar^{k-1}$ produces the same series

$$= \underset{\substack{\uparrow \\ k=1}}{ar^0} + \underset{\substack{\uparrow \\ k=2}}{ar^{2-1}} + \underset{\substack{\uparrow \\ k=3}}{ar^2} + \cdots$$

how do we find the $\overbrace{n^{th} \text{ partial sum}}^{\text{sum of first } n \text{ terms}}$ of a geometric series

$$\frac{1}{4} + \frac{1}{12} + \frac{1}{36} + \frac{1}{108} + \frac{1}{324} + \frac{1}{972} + \cdots$$

$$S_1 = \frac{1}{4}$$

$$S_2 = \frac{1}{4} + \frac{1}{12}$$

$$S_3 = \frac{1}{4} + \frac{1}{12} + \frac{1}{36}$$

$$\vdots$$

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \dots$$

$$S_1 = a$$

$$S_2 = a + ar^1$$

$$S_3 = a + ar + ar^2$$

$$S_4 = a + ar + ar^2 + ar^3$$

n^{th} partial sum ends with ar^{n-1}

$\vdots$

$$S_n = a + ar + ar^2 + \dots + ar^{n-1} \quad - 1$$

mutiply by r

$$rS_n = ar + ar^2 + ar^3 + \dots + ar^n \quad - 2$$

$1 - 2$

$$S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a - ar^n$$

$$\boxed{S_n = \frac{a - ar^n}{1-r}}$$

this allows us to find n^{th} partial sum knowing a, r

so, $\frac{1}{4}+\frac{1}{12}+\frac{1}{36}+\cdots = \sum_{k=0}^{\infty} \frac{1}{4}\left(\frac{1}{3}\right)^k$

7^{th} partial sum: $S_7 = \frac{a-ar^7}{1-r} = \frac{\frac{1}{4}\left(1-\left(\frac{1}{3}\right)^7\right)}{1-\frac{1}{3}}$

$$= \cdots = \boxed{\frac{1093}{2916}}$$

if we keep adding terms, what happens to S_n?

in other words, $\lim_{n\to\infty} S_n = ?$ (series converges if $\lim_{n\to\infty} S_n$ exists)

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} \underbrace{\lim_{n\to\infty}(1-r^n)}$$

exists only if $\lim_{n\to\infty} r^n$ exists

and that happens when $|r| < 1$

or $-1 < r < 1$

if $|r|<1$

then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \dfrac{a(1-\cancel{r^n})}{1-r} = \boxed{\dfrac{a}{1-r}}$ this is the sum of the geometric series

try this on $\displaystyle\sum_{k=0}^{\infty} \frac{1}{4}\underbrace{\left(\frac{1}{3}\right)^k}_{|r|<1} = \frac{1}{4}+\frac{1}{12}+\frac{1}{36}+\cdots = \frac{a}{1-r} = \frac{\frac{1}{4}}{1-\frac{1}{3}} = \frac{\frac{1}{4}}{\frac{2}{3}} = \boxed{\frac{3}{8}}$

if we keep adding terms, the sum gets closer to $\frac{3}{8}$

the sum formula also lets us convert repeating decimals into fractions

<u>example</u> $1.252525252525\ldots$

$$= 1 + \underbrace{0.25 + 0.0025 + 0.000025 + \cdots}_{\text{geometric series with } a = 0.25 \text{ and } r = 0.01 = \frac{1}{100}}$$

$$= 1 + \underbrace{\sum_{k=0}^{\infty} 0.25(0.01)^k}_{|r|<1 \text{ so a sum exists}}$$

$$= 1 + \frac{a}{1-r} = 1 + \frac{0.25}{1-0.01} = 1 + \frac{25}{99} = \boxed{\frac{124}{99}}$$

telescoping series

example $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k+11)(k+12)}$

$$= \frac{1}{12\cdot13} + \frac{1}{13\cdot14} + \frac{1}{14\cdot15} + \frac{1}{15\cdot16} + \cdots$$

clearly not geometric

does it converge?

partial fraction: $\displaystyle\frac{1}{(k+11)(k+12)} = \frac{1}{k+11} - \frac{1}{k+12}$

series becomes $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{k+11} - \frac{1}{k+12}\right)$

$$= \underbrace{\left(\frac{1}{12} - \cancel{\frac{1}{13}}\right)}_{k=1} + \underbrace{\left(\cancel{\frac{1}{13}} - \cancel{\frac{1}{14}}\right)}_{k=2} + \underbrace{\left(\cancel{\frac{1}{14}} - \frac{1}{15}\right)}_{k=3} + \cdots$$

only the first and last numbers survive

$$\left.\begin{aligned} S_3 &= \frac{1}{12} - \frac{1}{15} \\ S_4 &= \frac{1}{12} - \frac{1}{16} \\ S_5 &= \frac{1}{12} - \frac{1}{17} \end{aligned}\right\} S_n = \frac{1}{12} - \frac{1}{12+n}$$

does this series converge? $(n \to \infty)$

yes, it converges to $\frac{1}{12}$

the starting index does NOT affect the convergence of an infinite series

for example, we know $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$ converges (geometric with $a=1$, $r=\frac{1}{2}$)

$$= \underbrace{\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right)}_{\sum_{k=0}^{3} \left(\frac{1}{2}\right)^k} + \underbrace{\left(\frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots\right)}_{\sum_{k=4}^{\infty} \left(\frac{1}{2}\right)^k} = 2$$

this infinite series
must also converge
(cannot go to ∞)

so, if $\sum_{k=0}^{\infty} a_k$ converges, then $\sum_{k=17}^{\infty} a_k$ also converges (true for ANY infinite series, not just geometric)

also, $\sum_{k=0}^{\infty} (a_k + b_k) = \sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k$

converges only
if BOTH
$\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ converge