10.4 The Divergence and Integral Tests

we will learn various tests to see if an infinite series converges

last time: geometric series $\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots$ converges if $|r| < 1$

but what about for a non-geometric series, for example $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \sum_{\cancel{k=0}}^{\infty} \frac{\cancel{k}}{k^2}$

$$= \sum_{\cancel{k=2}}^{\infty} \frac{3}{2} + \frac{1}{k^2}$$

how ~~we~~ can we determine if such a series converges?

let's look at that question from the opposite point of view first

— what property must a convergent series have?

$\lim_{k \to \infty} S_k = L$ → must happen $\qquad$ S_k : k^{th} partial sum (sum of ~~the~~ first k terms)

$\sum_{k=1}^{\infty} a_k$

$S_1 = a_1$

$S_2 = a_1 + a_2$ $\qquad$ $S_2 - S_1 = a_2$

$S_3 = a_1 + a_2 + a_3$ $\qquad$ $S_3 - S_2 = a_3$

$S_4 = a_1 + a_2 + a_3 + a_4$

$\vdots$

$S_{k-1} = a_1 + a_2 + \cdots + a_{k-1}$

$S_k = a_1 + a_2 + \cdots + a_{k-1} + a_k$ $\qquad$ $S_k - S_{k-1} = a_k$

if series converges, then $\lim_{k\to\infty} S_k = L$ and $\lim_{k\to\infty} S_{k-1} = L$

but $\lim_{k\to\infty} (S_k - S_{k-1}) = \lim_{k\to\infty} a_k$

$$\underbrace{\lim_{k\to\infty} S_k}_{L} - \underbrace{\lim_{k\to\infty} S_{k-1}}_{L} = \lim_{k\to\infty} a_k$$

$\boxed{\text{so, if series } \sum_{k=1}^{\infty} a_k \text{ converges, then } \lim_{k\to\infty} a_k = 0}$ → if series converges, then terms eventually go to zero

this is called the Divergence Test

important: this if is a one-way if

if series converges, then $\lim_{k\to\infty} a_k = 0$

it is NOT correct to reverse the direction of that if, in general for example, $\sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k$ is a geometric series with $|r|<1$ so it converges

we see $\lim_{k\to\infty} (a_k) = \lim_{k\to\infty} \left(\frac{1}{3}\right)^k = \lim_{k\to\infty} \frac{1}{3^k} = 0$

on the other hand, $\sum_{k=0}^{\infty} \cos(k\pi) = 1-1+1-1+1-1+1-1+\ldots$

diverges since $\lim_{k\to\infty} S_k$ DNE

and we see $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \cos(k\pi)$ DNE

so, as we see that for a divergent series, $\lim_{k\to\infty} a_k \neq 0$

But, again, just because $\lim_{k\to\infty} a_k = 0$, it does <u>NOT</u> necessarily mean series converges

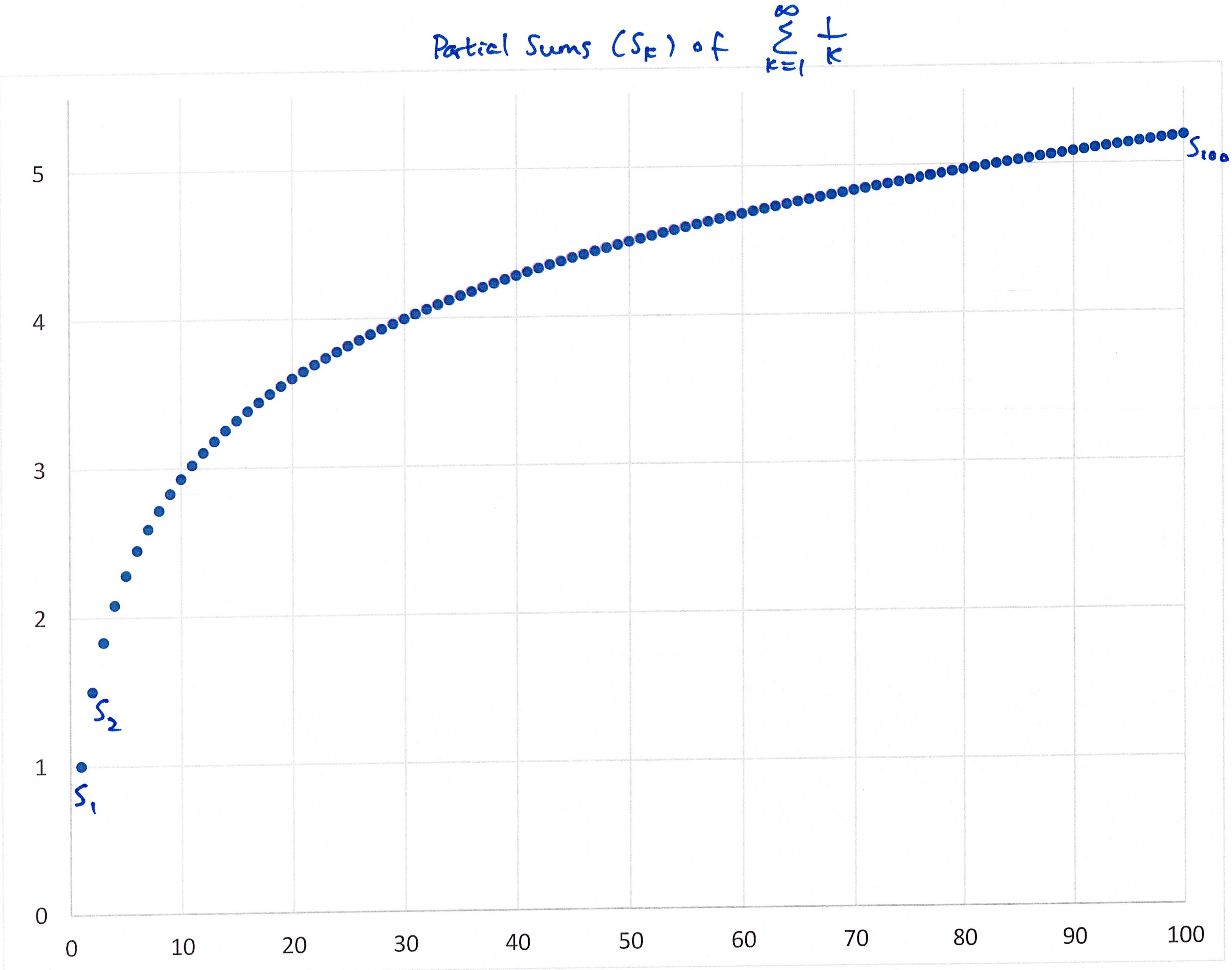
$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$ "Harmonic Series"

clearly, $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k} = 0$

But, this series actually diverges, even though $\lim_{k\to\infty} \frac{1}{k} = 0$

$$\left.\begin{aligned}
S_1 &= 1 \\
S_2 &= 1+\frac{1}{2} = 1.5 \\
S_3 &= 1+\frac{1}{2}+\frac{1}{3} = 1.833 \\
S_4 &= 2.0833 \\
&\vdots \\
S_{20} &= 3.5977 \\
&\vdots \\
S_{50} &= 4.4992 \\
&\vdots
\end{aligned}\right\}$$ S_k does not stop growing

$$\begin{aligned}
S_{100} &= 5.1874 \\
&\vdots \\
S_{500} &= 6.7928
\end{aligned}$$



Partial Sums (S_k) of $\sum_{k=1}^{\infty} \frac{1}{k}$
0
1
2
3
4
5
0
10
20
30
40
50
60
70
80
90
100
S_1
S_2
S_{100}

So, if $\lim_{k\to\infty} a_k = 0$, the series <u>might</u> converge.

How do we make sure? We need another test.

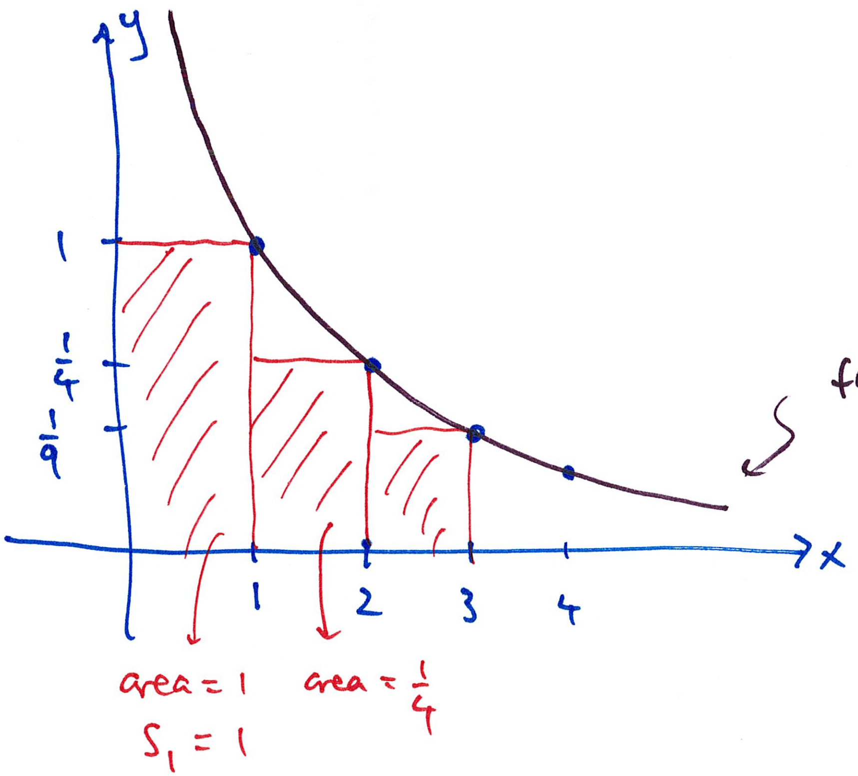
Today: <u>integral test</u>

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \underbrace{1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}}_{} + \cdots$$

$\lim_{k\to\infty} \frac{1}{k^2} = 0$, so this series <u>might</u> converge

treat these as points from $f(x) = \frac{1}{x^2}$ at $x = 1, 2, 3, \ldots$

let's graph these



now look at $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$

as a Riemann Sum

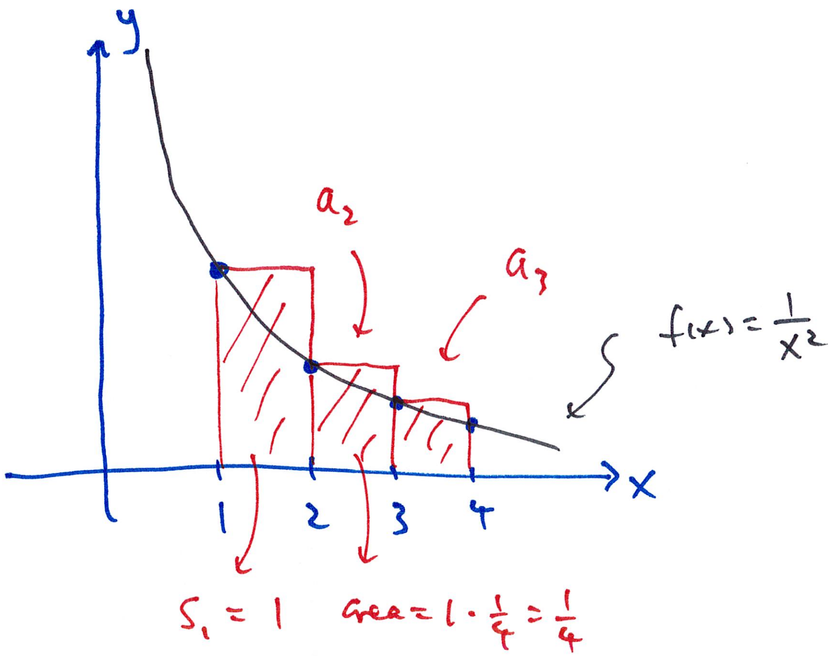
each partial sum is the total area of first boxes

we see from the picture

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k^2} \le \underbrace{1}_{\text{first box}} + \int_1^{\infty} \frac{1}{x^2}\,dx}$$

we will come back to this

now let's look at $\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots$ as a RIGHT Riemann Sum



this time, sum of boxes is bigger than area under curve

so,

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k^2} \geq \int_1^{\infty} \frac{1}{x^2}\,dx}$$

combining the two facts, we see

$$\int_1^{\infty} \frac{1}{x^2}\,dx \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \leq 1 + \int_1^{\infty} \frac{1}{x^2}\,dx$$

Integral Test

if the left and right bounds are NOT $\pm\infty$, then series converges

so, therefore, for $\sum_{k=1}^{\infty} a(k)$ to converge, the integral $\int_1^{\infty} f(x)\,dx$ must converge

where $f(x)$ is function where $a(k)$ is sampled

So does $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converge?

It passes the Divergence Test : $\lim_{k\to\infty} \frac{1}{k^2} = 0$

Integral Test : $\int_1^{\infty} \frac{1}{x^2}\,dx \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \leq 1 + \int_1^{\infty} \frac{1}{x^2}\,dx$

So, if $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges, then series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges

does $\int_1^{\infty} \frac{1}{x^2}\,dx$ converge? Yes, in fact, $\int_1^{\infty} \frac{1}{x^p}\,dx$ converges if $p > 1$

So, this means $\underbrace{\sum_{k=1}^{\infty} \frac{1}{k^p}}_{\text{"p-series"}}$ converges if $p > 1$

follow up to Integral Test : <u>P-series Test</u>

$\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$

$\sum_{k=1}^{\infty} \frac{1}{k}$ diverges , $\sum_{k=1}^{\infty} \frac{1}{k^{10}}$ converges the Integral Test is very general, but if the improper integral is hard to calculate, we might use a different ~~series~~ test (we will see a few more)