10.5 Comparison Tests

last time: Divergence Test: $\sum a_k$ ~~can~~ diverges if $\lim_{k\to\infty} a_k \neq 0$

But, even if $\lim_{k\to\infty} a_k = 0$, we cannot be sure if $\sum a_k$ converges

Integral Test: $\sum a_k$ converges if $\int_c^\infty a(k)\,dx$ converges

c ← doesn't have to be 1

p-series Test: $\sum \frac{1}{k^p}$ converges if $p > 1$

idea behind comparison test is to compare to a known series

for example, we know $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ converges

$$= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \quad \text{geometric with } |r| < 1$$

but $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \frac{1}{17} + \cdots$ is not a geometric series

is not a p-series

notice each term of this series is less than $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \frac{1}{17} + \cdots \leq \underbrace{1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots}_{\text{converges}} = 2$$

sum of geo. series $\frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$

so, $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{9}+\cdots$ also converges.

<u>example</u> $\sum_{k=1}^{\infty} \frac{1}{k^3+5}$

does it pass the Divergence Test? $\lim_{k\to\infty} \frac{1}{k^3+5} = 0$? yes, so the series <u>might</u> converge

Test more.

integral test? $\int_1^{\infty} \frac{1}{x^3+5}\,dx$ converges?

this is not an easy integral to perform

can we compare to a known series?

as $k\to\infty$, $\frac{1}{k^3+5}$ eventually resembles $\frac{1}{k^3}$

furthermore, we know $\frac{1}{k^3+5} \le \frac{1}{k^3}$ for $k \ge 1$

so this makes $\underbrace{\sum_{k=1}^{\infty} \frac{1}{k^3}}_{\substack{\text{p-series} \\ p=3>1 \\ \text{so converges}}}$ a good one to compare to since $\frac{1}{k^3+5} \le \frac{1}{k^3}$ for $k \ge 1$

we know $\sum_{k=1}^{\infty} \frac{1}{k^3+5} \le \sum_{k=1}^{\infty} \frac{1}{k^3} = L$ (because $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges)

so, $\sum_{k=1}^{\infty} \frac{1}{k^3+5} \le L$ so, $\sum_{k=1}^{\infty} \frac{1}{k^3+5}$ converges.

<u>example</u> $\sum_{k=1}^{\infty} \frac{k+1}{k^2}$

does it pass the Divergence Test? $\lim_{k\to\infty} \frac{k+1}{k^2} = 0$? yes, test more

Integral Test? $\int_1^{\infty} \frac{x+1}{x^2}\,dx$ is doable, but let's practice more with comparison test.

the convergence of a series depends on the behavior of the "tail" $(k \to \infty)$

$\sum_{k=1}^{\infty} \frac{k+1}{k^2}$ as k becomes large, $\frac{k+1}{k^2} \approx \frac{k}{k^2} \approx \frac{1}{k}$

the "tail" of $\sum_{k=1}^{\infty} \frac{k+1}{k^2}$ looks a lot like the tail of $\sum_{k=1}^{\infty} \frac{1}{k}$

So, we compare to $\sum_{k=1}^{\infty} \frac{1}{k}$

$$\sum_{k=1}^{\infty} \frac{k+1}{k^2} = 2 + \frac{3}{4} + \frac{4}{9} + \frac{5}{16} + \cdots$$

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

notice $\frac{k+1}{k^2} \geq \frac{1}{k}$ for $k \geq 1$

$\sum_{k=1}^{\infty} \frac{1}{k}$ diverges ($p = 1$, Harmonic series) $\rightarrow \sum_{k=1}^{\infty} \frac{1}{k} = \infty$

$$\sum_{k=1}^{\infty} \frac{k+1}{k^2} \geq \sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

so, $\sum_{k=1}^{\infty} \frac{k+1}{k^2} = \infty \rightarrow$ diverges if the terms of the series in question are $\boxed{\geq}$ those of a $\boxed{\text{divergent}}$ series,

then the series in question $\boxed{\text{diverges}}$

if the terms of the series in question are $\boxed{\leq}$ those of a $\boxed{\text{convergent}}$ series,

the the series in question $\boxed{\text{converges}}$

if the terms of the series in question are $\boxed{\leq}$ those of a $\boxed{\text{divergent}}$ series,

then we cannot conclusively say if the series converges

$$\sum_{k=1}^{\infty} a_k \quad \text{and} \quad a_k \leq \frac{1}{k} \quad \text{for} \quad k \geq 1$$

$$\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

$$\sum_{k=1}^{\infty} a_k < \infty \rightarrow \text{doesn't say anything useful}$$

(could sum to a number or a smaller ∞)

example $\sum_{k=1}^{\infty} \left(\frac{k}{2k+3}\right)^k$

does it pass the Divergence Test? $\lim_{k\to\infty} \left(\frac{k}{2k+3}\right)^k$

as k gets large $\frac{k}{2k+3} \approx \frac{k}{2k} \approx \frac{1}{2}$

looks a lot like $\lim_{k\to\infty} \left(\frac{1}{2}\right)^k = 0$

yes, passes DT

this also strongly suggest to compare to $\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$

a variation of the Comparison Test, called the Limit Comparison Test

compares the tail behavior of the series in question and ~~the~~ a known series

$\sum a_k$ is the one in question and $\sum b_k$ is known

- if $\lim_{k\to\infty} \frac{a_k}{b_k} = c$ (a number)

  then BOTH $\sum a_k$ and $\sum b_k$ converge

  or BOTH $\sum a_k$ and $\sum b_k$ diverge use it on this series

$$a_k = \left(\frac{k}{2k+3}\right)^k \quad \text{and} \quad b_k = \left(\frac{1}{2}\right)^k$$

$$\lim_{k\to\infty} \frac{\left(\frac{k}{2k+3}\right)^k}{\left(\frac{1}{2}\right)^k} = \lim_{k\to\infty} \left(\frac{\frac{k}{2k+3}}{\frac{1}{2}}\right)^k = \lim_{k\to\infty} \left(\underbrace{\frac{2k}{2k+3}}_{<1}\right)^k \neq \infty$$

limit is a number, so BOTH $\sum a_k$ and $\sum b_k$ converge or BOTH diverge

since we know $\sum b_k$ converges, we also know $\sum a_k$ converges

$$\sum_{k=1}^{\infty} \left(\frac{k}{2k+3}\right)^k \text{ converges}$$