10.6 Alternating Series

series with alternating signs

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$ Alternating Harmonic Series

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{2k-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$ ~~Leibnitz~~ Leibniz Series

$$= \frac{\pi}{4} \quad (\text{converges to } \tfrac{\pi}{4})$$

note the same series results if the power of (-1) is changed a bit

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{2k-1} = 1 - \frac{1}{3} + \frac{1}{5} + \cdots$$

generally, look for $(-1)^k$ kind of thing to identify alternating series

also, the "other part" is also nonnegative

$$\sum_{k=1}^{\infty} (-1)^{k+1} \underbrace{\frac{1}{k}}_{\geq 0} \qquad \sum_{k=1}^{\infty} (-1)^{k+1} a_k$$

An alternating series converges if the following conditions are met:

1) the magnitude of each term is <u>eventually</u> <u>nonincreasing</u>

$$a_{k+1} \le a_k \quad \text{for} \quad k \ge \text{some number}$$

2) $\lim\limits_{k \to \infty} a_k = 0$ (this is the Divergence Test)

this is called the <u>Alternating Series Test</u>

look at the alternating Harmonic (recall the "regular" Harmonic diverges)

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

it's easy to see $\lim\limits_{k \to \infty} a_k = \lim\limits_{k \to \infty} \frac{1}{k} = 0$

it's also easy to see that $\frac{1}{k}$ is nonincreasing (magnitude never gets bigger) for $k \ge 1$

So this series converges by the Alternating Series Test.

what does it converge to?

let's look at some partial sums

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

$$S_1 = 1$$

$$S_2 = 1 - \frac{1}{2} = \frac{1}{2} = 0.5$$

$$S_3 = 1 - \frac{1}{2} + \frac{1}{3} = 0.8333$$

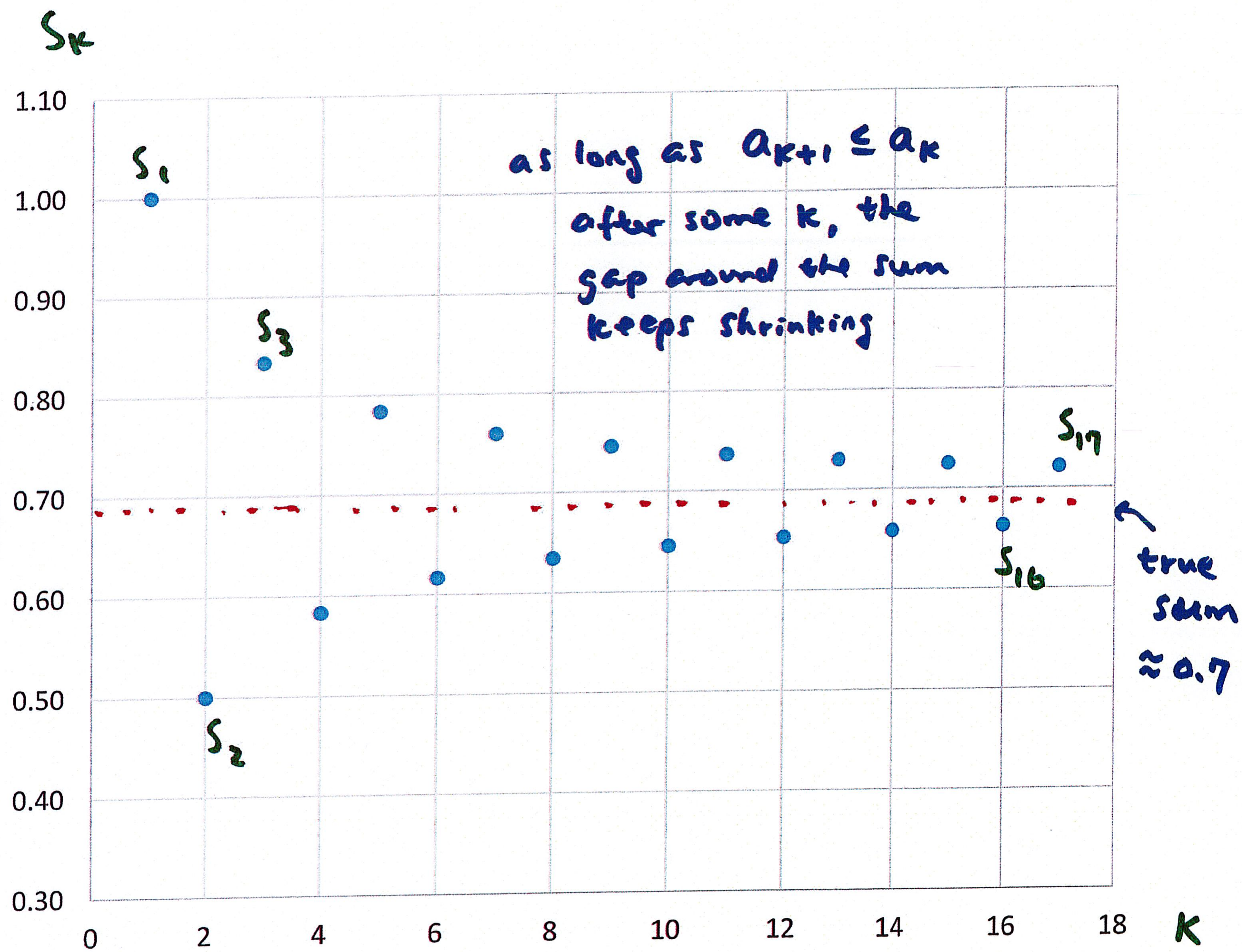
$$S_4 = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = 0.5833$$

$$S_5 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = 0.7833$$

note even when things are added back, the generally trend is still decreasing

S_n
1.0000
0.5000
0.8333
0.5833
0.7833
0.6167
0.7595
0.6345
0.7456
0.6456
0.6456
0.7365
0.6532
0.7301
0.6587
0.7254
0.6629
0.7217

Partial sums of $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$



example

Does $\sum_{k=0}^{\infty} \frac{(-1)^k}{k^6+9}$ converge?

does $a_k = \frac{1}{k^6+9}$ eventually ~~go~~ approach zero?

$\lim_{k \to \infty} \frac{1}{k^6+9} = 0$? yes, clearly.

is $a_k = \frac{1}{k^6+9}$ eventually nonincreasing?

one way is to look at its rate of change with respect to k

$$\frac{d}{dk}\left(\frac{1}{k^6+9}\right) = \frac{-6k^5}{(k^6+9)^2}$$

→ always ~~negative~~ nonpositive for $k = 0, 1, 2, 3, \ldots$

if negative, decreasing

if positive, increasing

we want to see it NOT increasing after some k

denominator is squared → always positive

so, $\frac{d}{dk}\left(\frac{1}{k^6+9}\right) \le 0$ for $k = 0, 1, 2, 3, \ldots$

always nonincreasing

this series passes the test, so will converge.

can we not just look at the first few terms to establish the "nonincreasing" part?

$$1 - 1 + \frac{1}{2} - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{3^2} + \frac{1}{4} - \frac{1}{4^5} + \cdots$$

a_k increases   decreases   increases

we can never be sure if this behavior is not hidden somewhere by just inspecting the terms

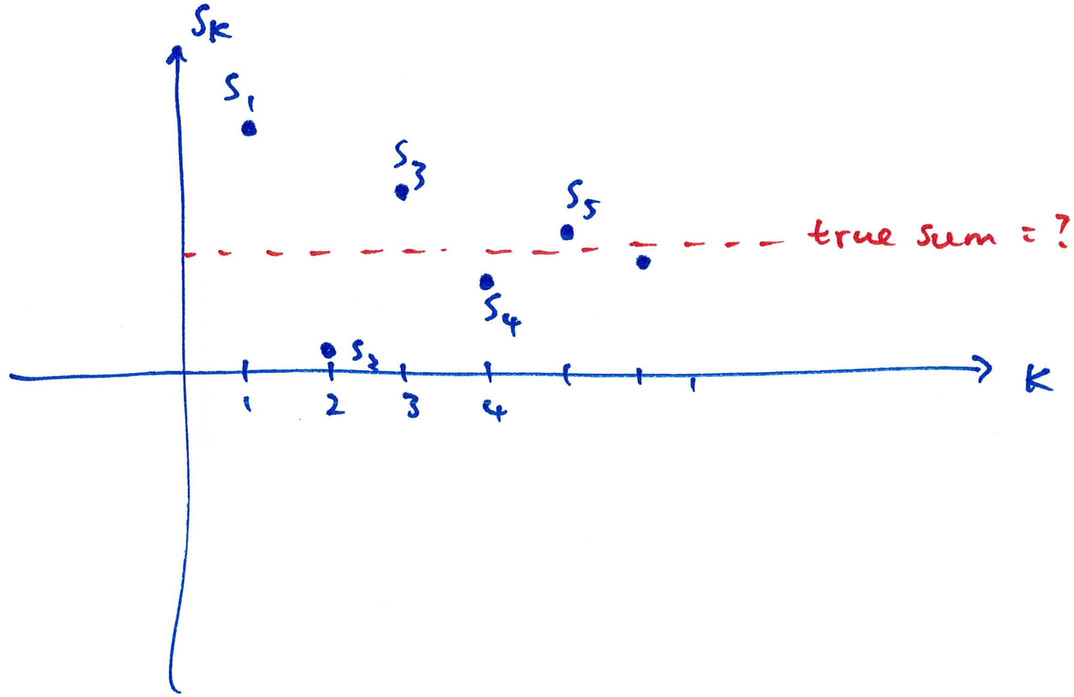
so, this series diverges, since a_k is never ~~no~~ always nonincreasing even though $\lim_{k \to \infty} a_k = 0$

we can estimate the sum of an alternating series to any accuracy we want,

if it converges

look at $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$, assume it converges

the graph of partial sums will look like



the true sum, S, is always between two consecutive partial sums

$$S_k \le S \le S_{k+1}$$

$S_k \le S \le S_{k+1}$

$0 \le |S - S_k| \le |S_{k+1} - S_k|$

note : $|S_{k+1} - S_k| = |a_{k+1}|$ why? note $\cancel{|S_1| = |a_1|}$

$\cancel{|S_2| = |S_1| + a_2}$

$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

$S_1 = 1$

$S_2 = 1 - \frac{1}{2}$

$S_3 = 1 - \frac{1}{2} + \frac{1}{3}$

$|S_3 - S_2| = \frac{1}{3} = |a_3|$

then, the above becomes

$$\boxed{0 \le |S - S_k| \le a_{k+1}}$$

this is how we estimate the sum S

example $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$

$$= \boxed{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}} + \underbrace{\boxed{\frac{1}{5}}}_{a_5} - \frac{1}{6} + \frac{1}{7} + \cdots$$

$\hookrightarrow S_4 = \frac{7}{12}$

the true sum S is : $0 \le |S - S_4| \le a_5$

so, $0 \le |S - \frac{7}{12}| \le \frac{1}{5} \longrightarrow \frac{7}{12} - \frac{1}{5} \le S \le \frac{7}{12} + \frac{1}{5}$

<u>example</u> $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$

How many terms do we need to add so that the partial sum is no more than 10^{-3} away from the true sum?

$0! = 1$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \underset{k=0}{1} - \underset{k=1}{1} + \underset{k=2}{\frac{1}{2}} - \frac{1}{6} + \cdots$$

after summing terms, we know S_k is no more than a_{k+1} away from the true sum $0 \le |S - S_k| \le a_{k+1}$

so, here we want $a_{k+1} \le 10^{-3}$ find k ~~that ma.~~

$$a_k = \frac{1}{k!}$$

$$a_{k+1} = \frac{1}{(k+1)!}$$

$$a_{k+1} \le 10^{-3} \rightarrow \frac{1}{(k+1)!} \le \frac{1}{1000} \rightarrow 1000 \le (k+1)!$$

this first happens at $k = 6$

so, we want to sum up to $k=6$ to ensure the accuracy if $\sum_{k=1}^{\infty} |a_k|$ converges, then we say $\sum_{k=1}^{\infty} a_k$

<u>converges absolutely</u> or is <u>absolutely convergent</u>

$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ is absolutely convergent because

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^k}{k^2} \right| = \sum_{k=1}^{\infty} \frac{1}{k^2} \text{ converges}$$

if a series is absolutely convergent, then it is convergent

if a series is convergent, but not absolutely, then it is <u>conditionally convergent</u>

e.g. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges but $\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges