10.7 Ratio and Root Tests

if $\sum_{k=1}^{\infty} |a_k|$ converges, then we say $\sum_{k=1}^{\infty} a_k$ converges absolutely

for example, $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} + \cdots$

and $\sum_{k=1}^{\infty} \left| \frac{(-1)^k}{k^2} \right| = \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$

converges (p-series $p > 1$)

so, we say $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ converges absolutely (and hence converges)

if $\sum_{k=1}^{\infty} a_k$ converges but $\sum_{k=1}^{\infty} |a_k|$ does not

then we say $\sum_{k=1}^{\infty} a_k$ converges conditionally

for example, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ converges

but $\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges

so, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges conditionally

Ratio Test

given $\sum_{k=1}^{\infty} a_k$, the series converges absolutely (and therefore converges)

if $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| < 1$

series diverges $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| > 1$

the test is inconclusive if $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = 1$

why? if $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| \cancel{<1} = r$

then this means for large k $|a_{k+1}| \approx r\,|a_k|$

that means each term is approximately r times the previous one

so, the tail of this series will look like a geometric

and it converges if $|r| < 1$

<u>example</u> $\displaystyle\sum_{k=1}^{\infty} \underbrace{k\left(\frac{1}{4}\right)^k}_{a_k}$

$$a_{k+1} = (k+1)\left(\frac{1}{4}\right)^{k+1}$$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)\left(\frac{1}{4}\right)^{k+1}}{k\left(\frac{1}{4}\right)^k}\right|$$

$$= \lim_{k\to\infty}\left|\underbrace{\frac{k+1}{k}}_{\text{goes to } 1}\cdot\left(\frac{1}{4}\right)\right| = \frac{1}{4} < 1$$

so $\displaystyle\sum_{k=1}^{\infty} k\left(\frac{1}{4}\right)^k$ converges (absolutely)

$\downarrow$

means the tail of $\displaystyle\sum_{k=1}^{\infty} k\left(\frac{1}{4}\right)^k$ looks like $\displaystyle\sum \left(\frac{1}{4}\right)^k$ which is a convergent geometric series ratio test is good with factorials

example $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{(2k+6)!} \rightarrow a_k$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{\frac{(k+1)!}{[2(k+1)+6]!}}{\frac{k!}{(2k+6)!}}\right| = \lim_{k\to\infty}\left|\frac{(k+1)!}{(2k+8)!}\cdot\frac{(2k+6)!}{k!}\right|$$

$$= \lim_{k\to\infty}\left|\frac{(k+1)!}{k!}\cdot\frac{(2k+6)!}{(2k+8)!}\right|$$

$$= \lim_{k\to\infty}\left|\frac{(k+1)\cancel{(k)}\cancel{(k-1)}\cancel{(k-2)}\cancel{\cdots}\cancel{(1)}}{\cancel{(k)}\cancel{(k-1)}\cancel{(k-2)}\cancel{\cdots}\cancel{(1)}}\cdot\frac{\cancel{(2k+6)}\cancel{(2k+5)}\cancel{(2k+4)}\cancel{\cdots}\cancel{(1)}}{(2k+8)(2k+7)\cancel{(2k+6)}\cancel{(2k+5)}\cancel{(2k+4)}\cancel{\cdots}\cancel{(1)}}\right|$$

$$= \lim_{k\to\infty}\left|\frac{k+1}{(2k+8)(2k+7)}\right| = 0 < 1$$

so series converges (absolutely)

<u>example</u> For what values of x does

$$\sum_{k=1}^{\infty} \frac{5x^k}{4k} \quad \text{converge?}$$

$$= \frac{5}{4}x + \frac{5}{8}x^2 + \frac{5}{12}x^3 + \frac{5}{16}x^4 + \cdots$$

apply Ratio Test

$$a_k = \frac{5x^k}{4k} \qquad a_{k+1} = \frac{5(x)^{k+1}}{4(k+1)}$$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{\frac{5x^{k+1}}{4(k+1)}}{\frac{5x^k}{4k}}\right| = \lim_{k\to\infty}\left|\frac{5x^{k+1}}{4(k+1)}\cdot\frac{4k}{5x^k}\right|$$

$$= \lim_{k\to\infty}\left|\frac{4k}{4k+4}\cdot\frac{5x^{k+1}}{5x^k}\right| = \lim_{k\to\infty}\left|\frac{4k}{4k+4}\cdot x\right| = |x| < 1$$

so x must be $|x| < 1$ for series to converge*, which means

$$\boxed{-1 < x < 1}$$

Root Test

given $\sum_{k=1}^{\infty} a_k$, it converges absolutely if $\lim_{k\to\infty} \sqrt[k]{|a_k|} < 1$

series diverges if $\lim_{k\to\infty} \sqrt[k]{|a_k|} > 1$

test is inconclusive if $\lim_{k\to\infty} \sqrt[k]{|a_k|} = 1$

why? $\lim_{k\to\infty} \sqrt[k]{|a_k|} = r$

then for large k $\sqrt[k]{|a_k|} \approx r$ so $|a_k| \approx r^k$

$\sum |a_k| \approx \sum r^k$ looks like a geo. series which converges if $|r| < 1$

Root Test is good if taking k^{th} root simplifies things (things with k in exponent)

example $\sum_{k=1}^{\infty} \frac{(k+1)^k}{k^{2k}}$ k in exponent : consider Root Test

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sqrt[k]{\frac{(k+1)^k}{(k^2)^k}} = \lim_{k\to\infty} \frac{k+1}{k^2} = 0 < 1$$

so series converges absolutely

what if we get 1 as limit in Ratio or Root test?

example $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$

note if we apply Ratio Test we get 1

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{\frac{k+1}{(k+1)^2+1}}{\frac{k}{k^2+1}}\right| = \lim_{k\to\infty}\left|\frac{k+1}{(k+1)^2+1}\cdot\frac{k^2+1}{k}\right|$$

$$= \lim_{k\to\infty}\left|\frac{k+1}{k}\cdot\frac{k^2+1}{(k+1)^2+1}\right| = \lim_{k\to\infty}\left|\underbrace{\frac{k+1}{k}}_{\to 1}\cdot\underbrace{\frac{k^2+1}{k^2+2k+2}}_{\to 1}\right| = 1 \quad \text{inconclusive}$$

now choose another test

comparison is good because as $k\to\infty$ $\frac{k}{k^2+1} \approx \frac{k}{k^2} \approx \frac{1}{k}$

limit comparison: $\lim_{k\to\infty}\frac{\frac{k}{k^2+1}}{\frac{1}{k}} = \lim_{k\to\infty}\frac{k}{k^2+1}\cdot k = \lim_{k\to\infty}\frac{k^2}{k^2+1} = 1$ (constant $0 < c < \infty$)

so BOTH converge/diverge

Since $\sum\frac{1}{k}$ diverges, $\sum\frac{k}{k^2+1}$ also diverges