10.8 Choosing a Convergence Test

Summary of Tests

Divergence Test : if $\lim_{k\to\infty} a_k = 0$ then series may converge, test more

if $\lim_{k\to\infty} a_k \neq 0$ then series diverges, STOP

Integral Test : $\sum_{k=1}^{\infty} a_k$ converges if $\int_1^{\infty} a_k(x)\,dx$ converges

p-series Test : $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$

Geometric series : $\sum_{k=0}^{\infty} ar^k$ converges if $|r| < 1$, sum $= \frac{a}{1-r}$

compare to

Direct Comparison Test : if $a_k \leq$ terms of a convergent series,
then $\sum a_k$ converges

if $a_k \geq$ terms of a divergent series,
then $\sum a_k$ diverges

Limit Comparison Test : $\sum a_k$ unknown , $\sum b_k$ known

$$\lim_{k\to\infty} \frac{a_k}{b_k} = c , \quad 0 < c < \infty$$, then BOTH $\sum a_k$ and $\sum b_k$ converge or diverge

Alternating Series Test : $\sum_{k=1}^{\infty} (-1)^k a_k$ converges if $\lim_{k\to\infty} a_k = 0$ $\underline{\underline{\text{AND}}}$ a_k is eventually nonincreasing

Ratio Test :

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| < 1 \to \text{series converges}$$

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| > 1 \to \text{series diverges}$$

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = 1 \to \text{inconclusive}$$

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \infty \to \text{inconclusive}$$

Root Test :

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} < 1 \quad \text{converges}$$

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = 1 \text{ or } \infty$$

inconclusive

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} > 1 \quad \text{diverges}$$

example $\displaystyle\sum_{k=1}^{\infty} \frac{11k^5 - 4k^3 + 5k + 10}{12k^5 + k^2 - k + 110}$

BEFORE doing anything, perform the Divergence Test

$$\lim_{k\to\infty} \frac{11k^5 - 4k^3 + 5k + 10}{12k^5 + k^2 - k + 110} = \frac{11}{12} > 0 \quad \text{so diverges}$$

example $\displaystyle\sum_{k=1}^{\infty} \left(1 + \frac{1}{k}\right)^k$

again, $\displaystyle\lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k \neq 0 \quad (= e)$

so, diverges

example $\displaystyle\sum_{k=1}^{\infty} \frac{1+\sin 9k}{k^2}$

does it pass the Divergence Test? yes, $\displaystyle\lim_{k\to\infty} \frac{1+\sin 9k}{k^2} = 0$

So, test more.

comparison is good here, since $0 \le 1+\sin 9k \le 2$

so a direct comparison to $\dfrac{2}{k^2}$ is appropriate

since $0 \le \dfrac{1+\sin 9k}{k^2} \le \dfrac{2}{k^2}$

and since $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k^2}$ is convergent (p-series w/ $p=2$)

so $\displaystyle\sum_{k=1}^{\infty} \frac{1+\sin 9k}{k^2} \le \sum_{k=1}^{\infty} \frac{2}{k^2}$ so is convergent.

what if we do a limit comparison to $\frac{2}{k^2}$ or $\frac{1}{k^2}$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{1+\sin 9k}{k^2}}{\frac{1}{k^2}} = \lim_{k\to\infty} (1+\sin 9k)$$ between 0 and 2

0 is problematic by itself

the limit is saying the tail of

$\sum \frac{1+\sin 9k}{k^2}$ is always no more than

twice of tail of $\frac{1}{k^2}$

constant multiple does not affect convergence, so we get same conclusion here.

example $\displaystyle\sum_{k=2}^{\infty} \frac{5}{k(\ln k)^7}$

does it pass the Div. Test? Yes, test more.

let's try integral test: does $\displaystyle\int_2^{\infty} \frac{5}{x(\ln x)^7}\,dx$ converge?

$$\lim_{b\to\infty} \int_2^{b} \frac{5}{x(\ln x)^7}\,dx$$

$u = \ln x$

$dx = \frac{1}{x}\,dx$

$$= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{5}{u^7}\,du \ne \infty$$

the number is not important

greater than 1
so converges

integral converges, so the series also converges.

$$\sum_{k=2}^{\infty} \frac{5}{k(\ln k)^7}$$

if we can show than $\frac{5}{(\ln k)^7} > 1$ for $k > 2$

then we can say $\frac{5}{k(\ln k)^7} > \frac{1}{k}$

but that's not true, so this reason shows

$\frac{5}{k(\ln k)^7} < \frac{1}{k}$ but $\sum \frac{1}{k}$ diverges so this test is also inconclusive example $\sum_{k=1}^{\infty} \left(\sqrt{16k^4+1} - 4k^2 \right)$

does it pass Div. Test? maybe? $\lim_{k\to\infty} \left(\sqrt{16k^4+1} - 4k^2 \right) = \infty - \infty$

$= ?$

to find that limit, we need l'Hospital's Rule

but is it worth the effort?

we can always pick an appropriate test and see what it says

maybe changing the form helps

$$\frac{\sqrt{16k^4+1} - 4k^2}{1} \cdot \frac{\sqrt{16k^4+1} + 4k^2}{\sqrt{16k^4+1} + 4k^2} = \frac{\cancel{16k^4} + 1 - \cancel{16k^4}}{\sqrt{16k^4+1} + 4k^2}$$

$$= \frac{1}{\sqrt{16k^4+1} + 4k^2}$$

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{16k^4+1}+4k^2}$$ looks like $$\sum \frac{1}{\sqrt{16k^4}+4k^2} = \sum \frac{1}{8k^2}$$

when k is large

so, we compare to $\sum \frac{1}{8k^2}$

$$\frac{1}{\sqrt{16k^4+1}+4k^2} \le \frac{1}{8k^2} \quad ?$$

$16k^4+1 > 16k^4$

so, $\sqrt{16k^4+1} > \sqrt{16k^4}$

so, $\sqrt{16k^4+1}+4k^2 > \sqrt{16k^4}+4k^2 = 8k^2$

so, $\frac{1}{\sqrt{16k^4+1}+4k^2} < \frac{1}{8k^2}$ so, converges because $\sum \frac{1}{8k^2}$ converges