11.1 Approximating Functions with Polynomials

Power series: $\sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$

a : center of power series

c_k : coefficient of k^{th} ~~deg~~ order term

a commonly used power series is the <u>Taylor Series</u>

goal: write functions (eg. e^x, $\sin x$) as a polynomial (Taylor series)

to find the Taylor series of a function $f(x)$, we match the function value and all its derivatives at $x=a$ to those of the power series

Taylor series of $f(x)$

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$$

match all derivatives and value at $x = a$

$$f(a) = c_0 + c_1\cancelto{0}{(a-a)} + c_2\cancelto{0}{(a-a)^2} + \cancelto{0}{\cdots} \rightarrow \boxed{f(a) = c_0} = 0!\, c_0$$

now match first deriv.

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

$$f'(a) = c_1 + 2\cancelto{0}{c_2(a-a)} + 3\cancelto{0}{c_3(a-a)^2} + \cancelto{0}{\cdots} \rightarrow \boxed{f'(a) = c_1} = 1!\, c_1$$

2nd deriv:

$$f''(x) = 2c_2 + 3\cdot 2\, c_3(x-a) + 4\cdot 3(x-a)^2 + \cdots$$

$$\boxed{f''(a) = 2c_2}$$

$$f'''(x) = 3\cdot 2\, c_3 + 4\cdot 3\cdot 2\, c_4(x-a) + \cdots$$

$$\boxed{f'''(a) = 3\cdot 2\, c_3}$$

$$f^{(4)}(x) = 4\cdot 3\cdot 2\cdot c_4 + \cdots$$

$$\boxed{f^{(4)}(a) = 4\cdot 3\cdot 2\cdot c_4}$$

predict: $f^{(k)}(a) = k!\, c_k$

so, $\boxed{\text{Taylor series coefficient: } c_k = \dfrac{f^{(k)}(a)}{k!}}$

So, the Taylor Series of $f(x)$ at $x=a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

behave like $f(x)$ "near" $x=a$

you have seen part of this before:

truncate after $k=1$, we get $f(x) \approx f(a) + f'(a)(x-a)$

Tangent Line approximation

So, think of Taylor Series as an extension of Tangent Line approx.

__example__ Find the Taylor Series of $f(x) = e^x$ near $x = a$, $a = 0$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \qquad \text{here, } a = 0$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$f(x) = e^x$

$f(0) = e^0 = 1$

$f'(x) = e^x$

$f'(0) = e^0 = 1$

$f''(x) = e^x$

$f''(0) = e^0 = 1$

$\vdots$

$f^{(k)}(x) = e^x$

$f^{(k)}(0) = e^0 = 1$

so, the Taylor series representation of e^x near $x=0$ is

$$f(x) = 1 + 1\cdot(x-0) + \frac{1}{2!}(x-0)^2 + \frac{1}{3!}(x-0)^3 + \frac{1}{4!}(x-0)^4 + \cdots$$

$$f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots = e^x$$

if we cut off after the k-th order, we get the k-th order <u>Taylor Polynomial</u> (P_k)

$$\left.\begin{aligned} P_0 &= 1 \\ P_1 &= 1+x \quad \text{(also the tangent line approx.)} \\ P_2 &= 1+x+\frac{1}{2}x^2 \\ P_3 &= 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3 \end{aligned}\right\} \approx e^x$$

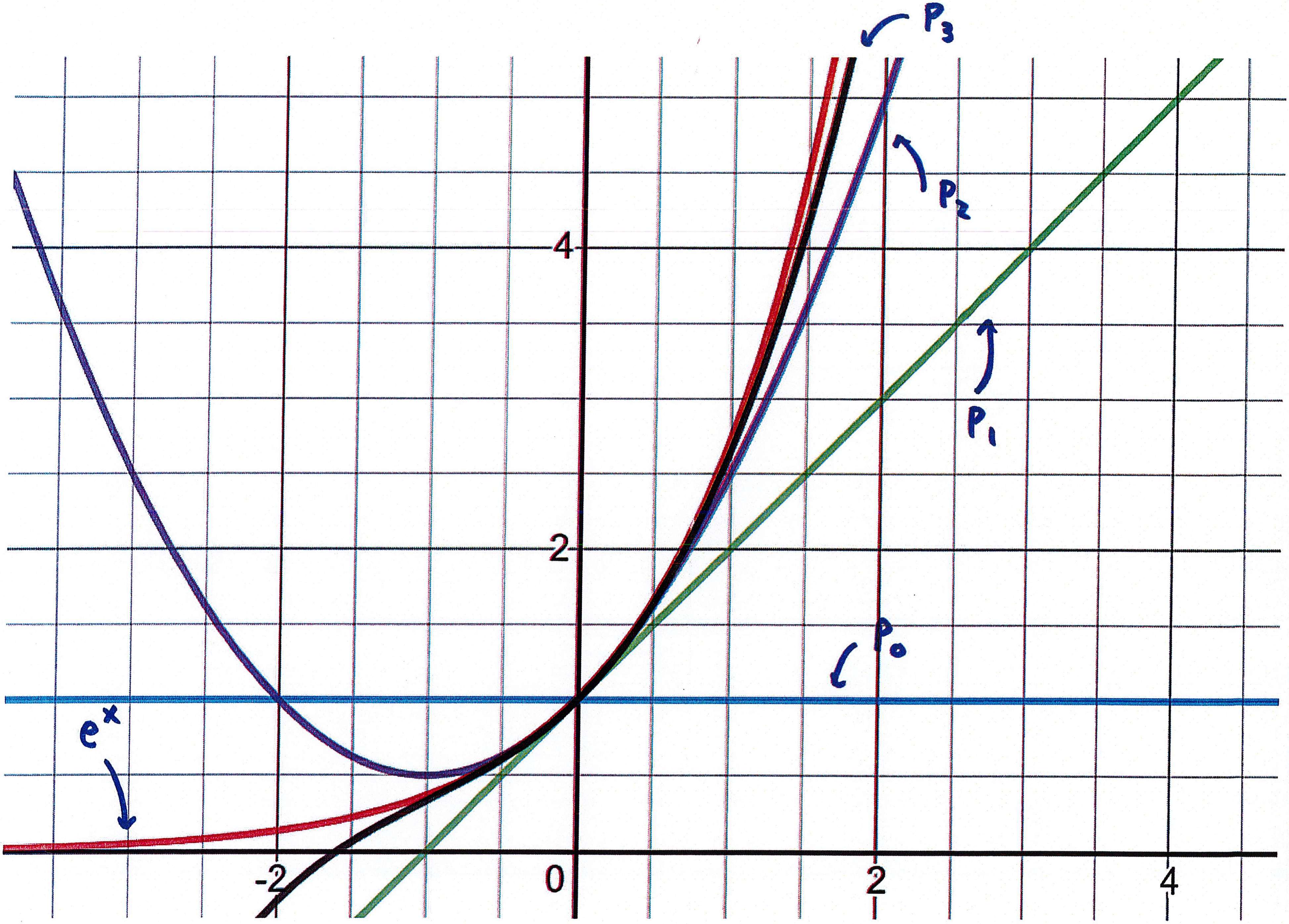
one way to use Taylor series is to find values like $e^{0.5}$. for example

it is approximately equal to (using P_3)

$$e^{0.5} \approx 1 + (0.5) + \frac{1}{2}(0.5)^2 + \frac{1}{6}(0.5)^3$$

$$\approx 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} = \frac{79}{48} \approx 1.6458$$

(true value of $e^{0.5}$ is 1.6487)



P3
P2
P1
P0
4
2
e^x
-2
0
2
4

example Find the 4^{th} order Taylor Polynomial of $f(x) = \cos(2x)$ near $x = a = \frac{\pi}{8}$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

we want to go up to $k = 4$, $a = \frac{\pi}{8}$

$$f(x) = f\left(\tfrac{\pi}{8}\right) + f'\left(\tfrac{\pi}{8}\right)\left(x - \tfrac{\pi}{8}\right) + \frac{f''\left(\frac{\pi}{8}\right)}{2!}\left(x - \tfrac{\pi}{8}\right)^2 + \frac{f'''\left(\frac{\pi}{8}\right)}{3!}\left(x - \tfrac{\pi}{8}\right)^3 + \frac{f^{(4)}\left(\frac{\pi}{8}\right)}{4!}\left(x - \tfrac{\pi}{8}\right)^4$$

$f(x) = \cos(2x) \longrightarrow f\left(\frac{\pi}{8}\right) = \frac{\sqrt{2}}{2}$

$f'(x) = -2\sin(2x) \longrightarrow f'\left(\frac{\pi}{8}\right) = -2 \cdot \frac{\sqrt{2}}{2} = -\sqrt{2}$

$f''(x) = -4\cos(2x) \longrightarrow f''\left(\frac{\pi}{8}\right) = -2\sqrt{2}$

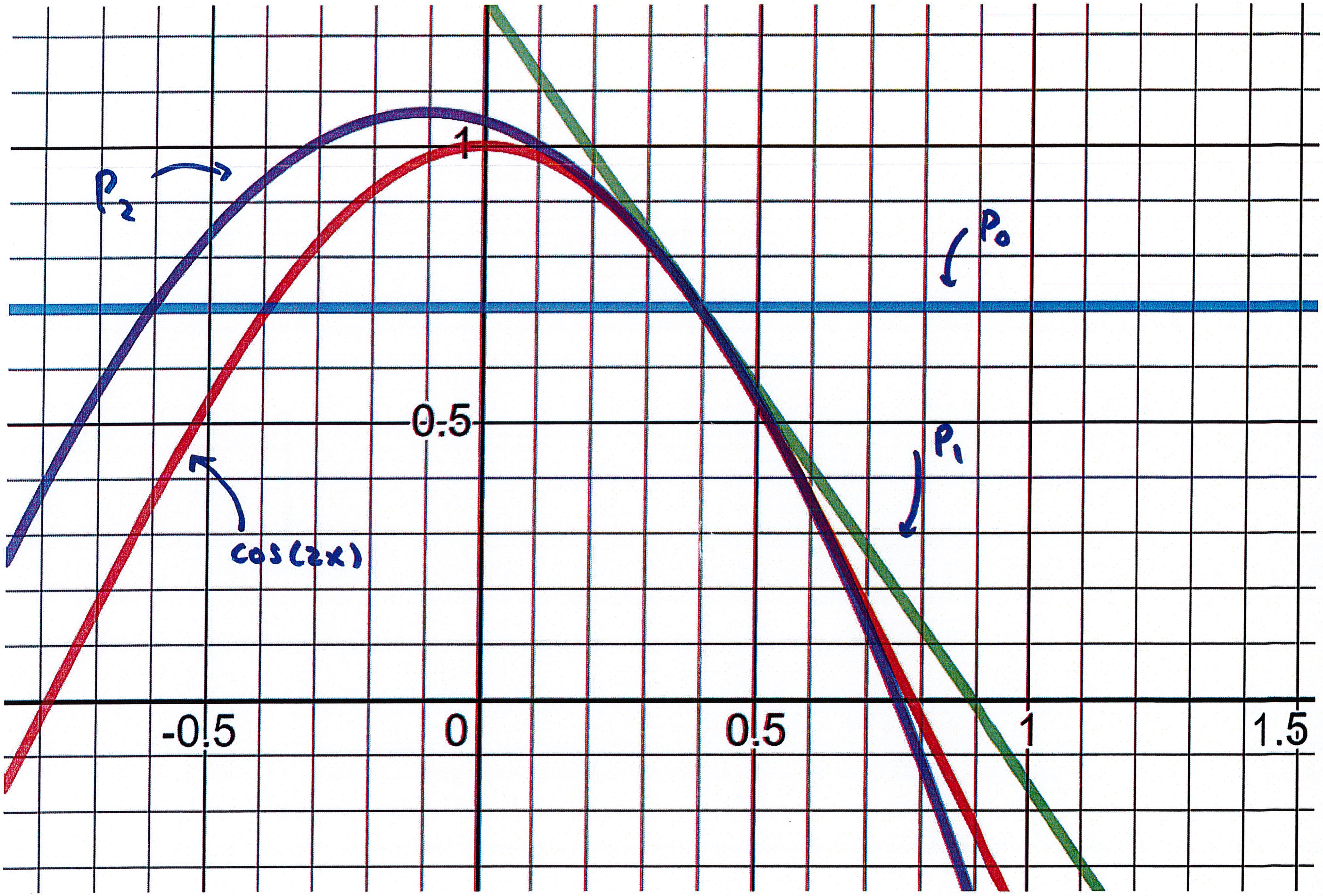
$f'''(x) = 8\sin(2x) \longrightarrow f'''\left(\frac{\pi}{8}\right) = 4\sqrt{2}$

$f^{(4)}(x) = 16\cos(2x) \longrightarrow f^{(4)}\left(\frac{\pi}{8}\right) = 8\sqrt{2}$

So, near $x = \frac{\pi}{8}$,

$$\cos(2x) \approx \underbrace{\underbrace{\underbrace{\underbrace{\underbrace{\frac{\sqrt{2}}{2}}_{P_0} - \sqrt{2}\left(x - \frac{\pi}{8}\right)}_{P_1} - \frac{2\sqrt{2}}{2!}\left(x - \frac{\pi}{8}\right)^2}_{P_2} + \frac{4\sqrt{2}}{3!}\left(x - \frac{\pi}{8}\right)^3}_{P_3} + \frac{8\sqrt{2}}{4!}\left(x - \frac{\pi}{8}\right)^4}_{P_4}$$

the higher the order, the better the approximation


1
0.5
-0.5
0
0.5
1
1.5
P_2
P_0
P_1
cos(2x)