11.1 Approx. Functions with Polynomials (continued)

Taylor series of $f(x)$ centered at $x=a$

is $$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

if we chop it off after, for example, $k=2$, then we got the 2nd-degree Taylor Polynomial

$$P_2 = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \approx f(x)$$ (approx. equal, but NOT exact)

the terms we throw away is the Remainder (error of approx.)

if $k=2$,

$$R_2 = \underbrace{\frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \cdots}_{\text{is an infinite series in itself}}$$

remainder

after 2nd order

is an infinite series in itself

which converges to a single term

what it converges to is given by the Taylor's Remaindor Theorem

remainder

in general, $f(x) = \underbrace{P_k(x)}_{\substack{k^{th}\text{ degree} \\ \text{Taylor polynomial}}} + R_k(x)$

it turns out that the remainder converges to

$$\boxed{R_k = \frac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1} \qquad \text{where} \quad a < c < x}$$

this is a generalized version of the Mean Value Theorem

c is usually not easy to find

but can we at least put a bound on $|R_k|$?

error of approx. $\rightarrow$

absolute error

$$|f(x) - P_k(x)| = |R_k(x)| = \left| \frac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1} \right|$$

in practice we don't find c, instead, we find the max of $|f^{(k+1)}(b)|$

on $a < b < x$, call $|f^{(k+1)}(b)| = M$

then we can bound the error as

$$|R_k(x)| \le \left| \frac{M}{(k+1)!}(x-a)^{k+1} \right|$$

<u>example</u> $f(x) = e^{-2x}$ $\quad a = 0$

$$
\begin{aligned}
f(x) &= e^{-2x} & f(a) &= f(0) = 1 &&= (-2)^0 \\
f'(x) &= -2e^{-2x} & f'(a) &= f'(0) = -2 &&= (-2)^1 \\
f''(x) &= 4e^{-2x} & f''(a) &= f''(0) = 4 &&= (-2)^2 \\
f'''(x) &= -8e^{-2x} & f''(a) &= f''(0) = -8 &&= (-2)^3 \\
&\vdots
\end{aligned}
$$

$$f^{K}(a) = (-2)^{k}$$

Taylor series :

$$e^{-2x} = f(x) = 1 - 2x + \frac{(-2)^2}{2!}x^2 + \frac{(-2)^3}{3!}x^3 + \frac{(-2)^4}{4!}x^4 + \cdots$$

now, let's use a 2nd order polynomial to approx. e^{-2x}

$$e^{-2x} \approx P_2(x) = 1 - 2x + \frac{(-2)^2}{2!}x^2$$

with a remainder (error)

$$R_2(x) = \frac{(-2)^3}{3!}x^3 + \frac{(-2)^4}{4!}x^4 + \cdots = \frac{f'''(c)}{3!}(x)^3 \qquad \overset{0}{a} < c < x$$

$$= \frac{-8e^{-2c}}{6}x^3 = -\frac{4}{3}e^{-2c}x^3$$

now let's use $P_2(x)$ to estimate $e^{-0.02}$ and estimate the error

$$e^{-2x} \approx P_2(x) = 1 - 2x + 2x^2$$

$$e^{-0.02} = e^{-2\underbrace{(0.01)}_{x}}$$

$$\approx P_2(0.01) \approx 1 - 2(0.01) + 2(0.01)^2 \approx \frac{4901}{5000}$$

what is the absolute error ?

it is exactly equal to $|R_2(0.01)| = \left| \frac{-4}{3} e^{-2c} (0.01)^3 \right|$

but we don't know c !

$$0 < c < 0.01$$

a x

can we at least bound $|e^{-2c}|$?

e^{-2x} is always decreasing, so max is at left ~~of~~ end of interval $\rightarrow$ $x = 0$

so, $|e^{-2c}| \leq |e^{-2(0)}| = 1$

so, $|R_2(0.01)| \leq \left| -\frac{4}{3} \cdot 1 \cdot (0.01)^3 \right| = \frac{4}{3{,}000{,}000}$

our approx. of $e^{-0.02} \approx \frac{4901}{5000}$ and it is no more than $\frac{4}{3{,}000{,}000}$ off

<u>example</u> $f(x) = \tan x \qquad a = 0$

use P_2 to estimate $\tan(0.5)$ and find the maximum absolute error

in P_2, we need up to $f''(x)$

$$P_2 = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$f(x) = \tan x \qquad f(0) = 0$

$f'(x) = \sec^2 x \qquad f'(0) = 1$

$f''(x) = 2\sec x \cdot \sec x \cdot \tan x$

$= 2\sec^2 x \cdot \tan x$

$f'''(x) = 2\sec^2 x\,(2\tan^2 x + \sec^2 x)$

$f''(0) = 0$

we need $f'''(x)$ in estimate of R_2

$P_2(x) = x \approx \tan(x)$ if we say "near" $a = 0$

$\tan(0.5) \approx 0.5$

how good (or bad) is the estimation?

$$|R_2(0.5)| = \left| \frac{f'''(c)}{3!}(0.5)^3 \right|$$

$f'''(x)$, here, is again monotonic function

so its maximum is at ends of interval $\quad 0 \le x \le 0.5$

$f'''(0) = 2$

$f'''(0.5) = 2\sec^2(0.5)\,(2\tan^2(0.5) + \sec^2(0.5))$

if we don't have a calculator, how can we estimate this?

trig functions "like" special values e.g. π, $\frac{\pi}{3}$, $\frac{\pi}{6}$, $\frac{\pi}{2}$, $\frac{\pi}{4}$

is $0.5 \approx$ one of those?

if we use $\pi \approx 3$, then $\frac{\pi}{6} \approx 0.5$

$$f'''\left(\frac{\pi}{6}\right) = 2\sec^2\left(\frac{\pi}{6}\right)\left(2\tan^2\left(\frac{\pi}{6}\right) + \sec^2\left(\frac{\pi}{6}\right)\right) = \frac{16}{3}$$

so, max of $f'''(x)$ on $0 \le x \le 0.5$ is about $\frac{16}{3}$ which replaces $f'''(c)$ in $|R_2(0.5)| = \left|\frac{f'''(c)}{3!}(0.5)^3\right|$

$$\le \left|\frac{1}{3\cdot 2}\cdot\frac{16}{3}\cdot\left(\frac{1}{2}\right)^3\right| = \left|\frac{1}{6}\cdot\frac{16}{3}\cdot\frac{1}{8}\right| = \frac{1}{9}$$

true value of $\tan(0.5)$ is within $0.5 \pm \frac{1}{9}$