11.2 Properties of Power Series

power series: $\sum_{k=0}^{\infty} c_k (x-a)^k$

convergence, in general, depends on x

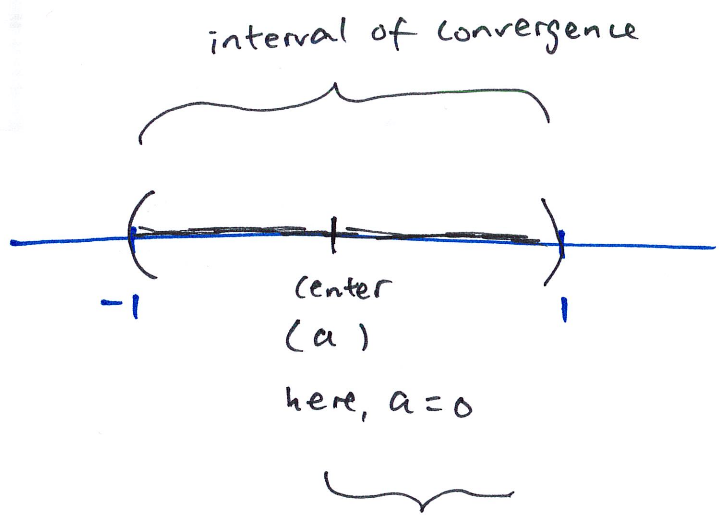
for example, the geometric series $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$

this converges if $|x| < 1$

for geometric series, we say the interval of convergence is $-1 < x < 1$

or $(-1, 1)$

number line:



half of interval
is the radius of convergence (here, it's 1)

for a general power series, the ends of the interval also need to be investigated example $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{\sqrt[3]{k}}$ find the radius and interval of convergence

use a test of our choice → typically the Ratio Test

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[3]{k}} (x-0)^k$$

so here, the center is $a=0$

now apply the Ratio Test

$$\lim_{k\to\infty} \left| \frac{\dfrac{(-1)^{k+1} x^{k+1}}{\sqrt[3]{k+1}}}{\dfrac{(-1)^k x^k}{\sqrt[3]{k}}} \right| = \lim_{k\to\infty} \left| \frac{(-1)^{k+1} x^{k+1}}{\sqrt[3]{k+1}} \cdot \frac{\sqrt[3]{k}}{(-1)^k x^k} \right|$$

$$= \lim_{k\to\infty} \left| (-1) \underbrace{\sqrt[3]{\frac{k}{k+1}}}_{1} \cdot x \right| = |x| < 1$$

so, the interval of convergence is $-1 < x < 1$ and possibly also at

$x = 1$ and /or $x = -1$

Ratio Test is inconclusive

let's find out what happens at $x = -1$

$$\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{\sqrt[3]{k}}$$

at $x = -1$, we get $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k (-1)^k}{\sqrt[3]{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/3}}$

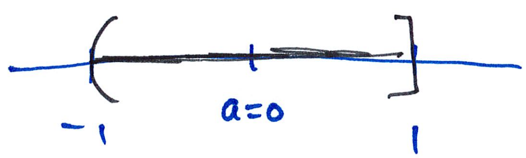
p-series
diverges because $p < 1$

now try $x = 1$

at $x = 1$, we get $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{1/3}}$

alt. series
converges because $\frac{1}{k^{1/3}}$ is nonincreasing and eventually approaches 0.

So, for this series, the interval of convergence is $(-1, 1]$



radius of convergence is 1

(notice the radius of convergence is not affected by the end behaviors)

example $\displaystyle\sum_{k=1}^{\infty} (kx)^k = \sum_{k=1}^{\infty} k^k \cdot x^k = \sum_{k=1}^{\infty} k^k \cdot (x-0)^k$ center : $a=0$

apply Ratio Test

$$\lim_{k\to\infty} \left| \frac{(k+1)^{k+1}\, x^{k+1}}{k^k\, x^k} \right| = \lim_{k\to\infty} \left| \underbrace{\left(\frac{k+1}{k}\right)^k}_{\text{some finite number}} \cdot \underbrace{(k+1)}_{\infty} \cdot (x) \right| = \infty \quad \text{unless } x=0$$

this means the series diverges (because limit is not less than 1)

unless $x=0$ (where the center is)

it does converge at $x=0$: $\displaystyle\sum_{k=1}^{\infty} (kx)^k = \sum_{k=1}^{\infty} 0 = 0+0+0+0\cdots = 0$

$a=0$

interval: $x=0$

radius : $R=0$

sometimes we need to come up with the summation notation first

<u>example</u>

$$x - \frac{x^3}{4} + \frac{x^5}{9} - \frac{x^7}{16} + \cdots$$

patterns: numerator → alternating, x to increasing odd powers

denominator → something squared

$$= \frac{x^1}{1} - \frac{x^3}{2^2} + \frac{x^5}{3^2} - \frac{x^7}{4^2} + \frac{x^9}{5^2} - \cdots$$

$k=1 \quad k=2 \quad k=3 \quad k=4 \quad k=5$

now decide starting k

here, $k=1$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{2k-1}}{k^2}$$

powers series as functions

we know $\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \cdots = \dfrac{a}{1-r}$ if $|r| < 1$

notice if $a=1$, $r=x$ we get

$$\frac{a}{1-r} = \boxed{\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots} \quad |x|<1$$

this function is now written as a power series

(we found its Taylor series w/o taking derivatives!)

we can modify it to find series for similar-looking functions

$$\frac{1}{1+x} = \underbrace{\frac{1}{1-(-x)}}_{\substack{\text{replace } x \text{ with } -x \\ \text{in } \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k}} = \sum_{k=0}^{\infty} (-x)^k = \sum_{k=0}^{\infty} (-1)^k x^k = 1 - x + x^2 - x^3 + x^4 - \cdots$$

$|-x| = |x| < 1$

$$\frac{x^2}{1+x^2} = x^2 \cdot \frac{1}{1+x^2}$$

$$= x^2 \cdot \frac{1}{1-(-x^2)} = x^2 \cdot \sum_{k=0}^{\infty} (-x^2)^k = x^2 \cdot \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

change x to $-x^2$

in $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

$$= x^2\left(1 - x^2 + x^4 - x^6 + x^8 - \cdots\right)$$

$$= x^2 - x^4 + x^6 - x^8 + x^{10} - \cdots = \sum_{k=0}^{\infty} (-1)^k x^{2k+2}$$

$|-x^2| < 1$

the "1" in $\frac{1}{1-x}$ in denominator is important

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$\frac{1}{2-x} = \frac{1}{2\left(1-\frac{x}{2}\right)} = \frac{1}{2} \cdot \frac{1}{1-\left(\frac{x}{2}\right)} = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k \qquad \left|\frac{x}{2}\right| < 1$$

$$|x| < 2$$

must be 1 to reuse series

go back to

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

change x to $\frac{x}{2}$