11.2 Properties of Power Series (continued)

last time: $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ $\quad |x| < 1$

modify for series like $\frac{1}{1+x^2}$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

change x to $-x^2$ in $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

$$= \sum_{k=0}^{\infty} (-x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

$|-x^2| < 1$

$\hookrightarrow |x| < 1$

we can also differentiate or integrate "model series" for even more series.

example Write the power series representation for $g(x) = \frac{1}{(1+x^2)^2}$

we have a series for $\frac{1}{1-x}$ and $\frac{1}{1+x^2}$.

the square must be handled differently

note $\frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{-2x}{(1+x^2)^2} = -2x \cdot \frac{1}{(1+x^2)^2} = -2x \cdot g(x)$

we want $g(x)$: $g(x) = -\frac{1}{2x} \cdot \frac{d}{dx}\underbrace{\left(\frac{1}{1+x^2}\right)}_{\text{we can get a power series for this}}$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

$$\frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} (-1)^k x^{2k}\right) = \frac{d}{dx}\left(1 - x^2 + x^4 - x^6 + x^8 - \cdots\right)$$

$$= -2x + 4x^3 - 6x^5 + 8x^7 - 10x^9 + 12x^{11} - \cdots$$

$$-\frac{1}{2x}\frac{d}{dx}\left(\frac{1}{1+x^2}\right) = -\frac{1}{2x}\left(-2x + 4x^3 - 6x^5 + 8x^7 - 10x^9 + 12x^{11} - \cdots\right)$$

$= 1 - 2x^2 + 3x^4 - 4x^6 + 5x^8 - 6x^{10} + 7x^{12} - \dots$

now put into summation notation

pattern: alternating, even powers of x, coefficients go up by 1

decide what k to start with: $k=1$

$$\underset{k=1}{1} \; \underset{k=2}{- 2x^2} \; \underset{k=3}{+ 3x^4} \; \underset{k=4}{- 4x^6} \; \underset{k=5}{+ 5x^8} - 6x^{10} + 7x^{12} - \dots$$

$$= \boxed{\sum_{k=1}^{\infty} (-1)^{k+1} \cdot k \cdot x^{2k-2}}$$

differentiation and integration do NOT affect the radius of convergence

BUT, <u>may</u> change the convergence at the ends of the interval of convergence

so, in $\dfrac{1}{1-x} = \displaystyle\sum_{k=0}^{\infty} x^k$, we need $|x| < 1$

and in $\dfrac{1}{(1+x^2)^2}$ which came from deriv. of $\dfrac{1}{1-x}$

the interval remains $|x| < 1 \Leftrightarrow -1 < x < 1$

BUT, the ~~con~~ series may or may not converge at $x = 1$ or -1

example $\ln\sqrt{16-x^2}$

build using $\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

$$\ln(16-x^2)^{1/2} = \frac{1}{2}\ln(16-x^2)$$

use

notice $\dfrac{d}{dx}\left[\frac{1}{2}\ln(16-x^2)\right] = \dfrac{1}{2}\cdot\dfrac{-2x}{16-x^2} = -x\cdot\boxed{\dfrac{1}{16-x^2}}$

$$\frac{1}{16-x^2} = \frac{1}{16(1-\frac{x^2}{16})} = \frac{1}{16}\cdot\frac{1}{1-\frac{x^2}{16}} = \frac{1}{16}\cdot\boxed{\frac{1}{1-(\frac{x}{4})^2}}$$

power series?

$$= \frac{1}{16}\sum_{k=0}^{\infty}\left(\frac{x^2}{16}\right)^k = \frac{1}{16}\sum_{k=0}^{\infty}\frac{x^{2k}}{4^{2k}}$$

$$-x\cdot\frac{1}{16-x^2} = -x\cdot\frac{1}{16}\sum_{k=0}^{\infty}\frac{x^{2k}}{4^{2k}} = \frac{d}{dx}\Big[\overbrace{\frac{1}{2}\ln(16-x^2)}^{\text{what we want}}\Big]$$

$$\ln\sqrt{16-x^2} = \frac{1}{2}\ln(16-x^2) = \int\left(\frac{-x}{16}\sum_{k=0}^{\infty}\frac{x^{2k}}{4^{2k}}\right)dx$$

$$= \int -\frac{x}{16}\left(1+\frac{x^2}{4^2}+\frac{x^4}{4^4}+\frac{x^6}{4^6}+\frac{x^8}{4^8}+\cdots\right)dx$$

$$= \int\left(-\frac{x}{4^2}-\frac{x^3}{4^4}-\frac{x^5}{4^6}-\frac{x^7}{4^8}-\frac{x^9}{4^{10}}-\cdots\right)dx$$

$$= -\frac{x^2}{2\cdot 4^2}-\frac{x^4}{4\cdot 4^4}-\frac{x^6}{6\cdot 4^6}-\frac{x^8}{8\cdot 4^8}-\frac{x^{10}}{10\cdot 4^{10}}-\frac{x^{12}}{12\cdot 4^{12}}-\cdots+C = \ln\sqrt{16-x^2}$$

what is C?

at $x=0$, everything on the left except C is 0

$0 + C = \ln 4 \rightarrow C = \ln 4$

so, $\ln\sqrt{16-x^2}$

$$= \ln 4 - \underbrace{\left(\overset{k=1}{\frac{x^2}{2\cdot 4^2}}+\overset{k=2}{\frac{x^4}{4\cdot 4^4}}+\overset{k=3}{\frac{x^6}{6\cdot 4^6}}+\overset{k=4}{\frac{x^8}{8\cdot 4^8}}+\cdots\right)}_{\text{write as summation, start at } k=1}$$

$$= \boxed{\ln 4 - \sum_{k=1}^{\infty}\frac{1}{2k}\left(\frac{x}{4}\right)^{2k}}$$

radius of convergence $= 1$

interval of convergence : $-1 < x < 1$
and maybe at $x=1$ and/or $x=-1$

example what function is represented by $\sum_{k=0}^{\infty} \frac{(x-2)^k}{3^{2k}}$?

only model series we know for now: $\frac{1}{1-x} = \boxed{\sum_{k=0}^{\infty} x^k}$ $\quad |x|<1$

$$\sum_{k=0}^{\infty} \frac{(x-2)^k}{3^{2k}} = \boxed{\sum_{k=0}^{\infty} \left[\frac{(x-2)}{3^2}\right]^k}$$

so, $\frac{x-2}{3^2}$ is the x in the model series

$$\text{so, } \sum_{k=0}^{\infty} \left[\frac{(x-2)}{3^2}\right]^k = \frac{1}{1-\left(\frac{x-2}{9}\right)} = \frac{9}{9-(x-2)} = \boxed{\frac{9}{11-x}}$$

interval of convergence?

$|x|<1$ in model series

in our series, "x" is $\frac{x-2}{3^2}$

$$\text{so, } \left|\frac{x-2}{9}\right| < 1 \rightarrow |x-2|<9 \rightarrow -9 < x-2 < 9$$

$$\boxed{-7 < x < 11}$$