11.3 Taylor Series

Taylor series: $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ $\qquad$ a : center

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

if $a=0$, the resulting series is often called the <u>Maclaurin Series</u>

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

<u>example</u> Find the Maclaurin Series for $f(x) = \sin x$

$f(x) = \sin x$ $\qquad$ $f(0) = 0$

$f'(x) = \cos x$ $\qquad$ $f'(0) = 1$

$f''(x) = -\sin x$ $\qquad$ $f''(0) = 0$ $\qquad$ even derivatives are zero

$f'''(x) = -\cos x$ $\qquad$ $f'''(0) = -1$ $\qquad$ odd " are 1 or −1

$f^{(4)}(x) = \sin x$ $\qquad$ $f^{(4)}(0) = 0$

pattern repeats

So, the Maclaurin series for $\sin x$ is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$= \underset{k=0}{x} - \underset{k=1}{\frac{1}{3!}x^3} + \underset{k=2}{\frac{1}{5!}x^5} - \underset{k=3}{\frac{1}{7!}x^7} + \underset{k=4}{\frac{1}{9!}x^9} + \cdots$$

$$= \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}}$$

Maclaurin series for $\sin x$

$\sin x$ behaves like this near $x = 0$

$$\sin x \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

interval of convergence?

Ratio Test :
$$\lim_{k\to\infty} \left| \frac{(-1)^{k+1} \dfrac{x^{2(k+1)+1}}{[2(k+1)+1]!}}{(-1)^k \dfrac{x^{2k+1}}{(2k+1)!}} \right| = \lim_{k\to\infty} \left| \frac{x^{2k+3}}{(2k+3)!} \cdot \frac{(2k+1)!}{x^{2k+1}} \right|$$

$$= \lim_{k\to\infty} \left| x^2 \cdot \frac{\cancel{(2k+1)}\cancel{(2k)}\cancel{(2k-1)}\cdots\cancel{(1)}}{(2k+3)(2k+2)\cancel{(2k+1)}\cancel{(2k)}\cancel{(2k-1)}\cdots\cancel{(1)}} \right| = 0$$

this means x can be anything, the ratio is always less than 1 interval of convergence : $(-\infty, \infty)$

radius of convergence : ∞

just like with $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, we can modify known series to find series of similar functions.

<u>example</u> $f(x) = x^3 \sin\left(\frac{x^2}{3}\right)$

we know $\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ $(-\infty, \infty)$

so, $\sin\left(\frac{x^2}{3}\right) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} \left(\frac{x^2}{3}\right)^{2k+1}$

$= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} \frac{x^{4k+2}}{3^{2k+1}}$ converges on $(-\infty, \infty)$

$x^3 \sin\left(\frac{x^2}{3}\right) = x^3 \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} \frac{x^{4k+2}}{3^{2k+1}}$ bring in 3 more in power of x (add 3)

$= \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} \frac{x^{4k+5}}{3^{2k+1}}} = \frac{x^5}{3} - \frac{x^9}{3!\,3^2} + \frac{x^{13}}{5!\,3^5} - \cdots$

Maclaurin series of common functions

↳ $a=0$ ANY other $a \to$ must do Taylor series the "proper" way (differentiation)

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k, \quad \text{for } |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^k x^k + \cdots = \sum_{k=0}^{\infty} (-1)^k x^k, \quad \text{for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for } |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{k+1} x^k}{k} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \quad \text{for } -1 < x \le 1$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k}, \quad \text{for } -1 \le x < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^k x^{2k+1}}{2k+1} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \quad \text{for } |x| \le 1$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

$$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k, \text{ for } |x| < 1 \text{ and } \binom{p}{k} = \frac{p(p-1)(p-2)\cdot \cdots \cdot (p-k+1)}{k!}, \quad \binom{p}{0} = 1$$

example $f(x) = \frac{1}{5+x^2}$

$$= \frac{1}{5\left(1+\frac{x^2}{5}\right)} = \frac{1}{5} \cdot \frac{1}{1+\left(\frac{x}{\sqrt{5}}\right)^2}$$

model series: $\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k \qquad |x|<1$

$$= \frac{1}{5} \sum_{k=0}^{\infty} (-1)^k \left[\left(\frac{x}{\sqrt{5}}\right)^2\right]^k$$

$$= \frac{1}{5} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{5^k} = \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{5^{k+1}}}$$

interval of convergence: $\left|\left(\frac{x}{\sqrt{5}}\right)^2\right| < 1$

$$\frac{x^2}{5} < 1$$

$$x^2 < 5$$

$$\boxed{-\sqrt{5} < x < \sqrt{5}}$$

example $f(x) = \sin x$ $\underbrace{a = \frac{\pi}{6}}_{\text{not } 0}$

$$f(x) = \sin x \qquad f\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$f'(x) = \cos x \qquad f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$f''(x) = -\sin x \qquad f''\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$

$$f'''(x) = -\cos x \qquad f'''\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$f^{(4)}(x) = \sin x \qquad f^{(4)}\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$f^{(5)}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$\sin x = \frac{1}{2} + \frac{\sqrt{3}/2}{1!}\left(x - \frac{\pi}{6}\right) - \frac{1/2}{2!}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}/2}{3!}\left(x - \frac{\pi}{6}\right)^3 + \frac{1/2}{4!}\left(x - \frac{\pi}{6}\right)^4 + \cdots$$

Summation notation?

note even derivatives are $\pm \frac{1}{2}$, alternating signs

odd " " $\pm \frac{\sqrt{3}}{2}$, " "

$\rightarrow$ split them up

$$= \left(\underset{k=0}{\frac{1}{2}} - \frac{1/2}{2!}\underset{k=1}{\left(x-\frac{\pi}{6}\right)^2} + \frac{1/2}{4!}\underset{k=2}{\left(x-\frac{\pi}{6}\right)^4} - \frac{1/2}{6!}\left(x-\frac{\pi}{6}\right)^6 + \cdots \right)$$ ← even powers

$$+ \left(\underset{k=0}{\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)} - \underset{k=1}{\frac{\sqrt{3}/2}{3!}}\left(x-\frac{\pi}{6}\right)^3 + \underset{k=2}{\frac{\sqrt{3}/2}{5!}}\left(x-\frac{\pi}{6}\right)^5 - \cdots \right)$$ ← odd powers

$$= \boxed{\sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{2(2k)!}\left(x-\frac{\pi}{6}\right)^{2k} + \sum_{k=0}^{\infty} (-1)^k \frac{\sqrt{3}}{2(2k+1)!}\left(x-\frac{\pi}{6}\right)^{2k+1}}$$

Taylor series of $\sin x$ at $a = \frac{\pi}{6}$

interval of convergence: $(-\infty, \infty)$ because $\sin x$ built at $a=0$ converges on $(-\infty, \infty)$ ~~so~~

changing a does <u>NOT</u> change that