11.4 Working with Taylor Series

Taylor series: $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

any function that is differentiable near $x=a$ has a Taylor series

which is a polynomial and is often easier to work with

if $a=0$, we usually just reuse the "model" series

Maclaurin series of common functions

↳ $a=0$ ANY other $a \to$ must do Taylor series the "proper" way (differentiation)

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k, \quad \text{for } |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^k x^k + \cdots = \sum_{k=0}^{\infty} (-1)^k x^k, \quad \text{for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for } |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{k+1} x^k}{k} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \quad \text{for } -1 < x \le 1$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k}, \quad \text{for } -1 \le x < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^k x^{2k+1}}{2k+1} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \quad \text{for } |x| \le 1$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

$$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k, \text{ for } |x| < 1 \text{ and } \binom{p}{k} = \frac{p(p-1)(p-2)\bullet\cdots\bullet(p-k+1)}{k!}, \quad \binom{p}{0} = 1$$

example $f(x) = \dfrac{9\tan^{-1}(x) - 9x + 3x^3}{5x^5}$, $a = 0$

very tedious to find the Taylor series by differentiation

from Table : $\tan^{-1}(x) = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dfrac{x^9}{9} - \cdots$

$9\tan^{-1}(x) = 9x - \dfrac{9x^3}{3} + \dfrac{9x^5}{5} - \dfrac{9x^7}{7} + \dfrac{9x^9}{9} - \cdots$

$9\tan^{-1}(x) - 9x = -\dfrac{9x^3}{3} + \dfrac{9x^5}{5} - \dfrac{9x^7}{7} + \dfrac{9x^9}{9} - \cdots$

$9\tan^{-1}(x) - 9x + 3x^3 = \dfrac{9x^5}{5} - \dfrac{9x^7}{7} + \dfrac{9x^9}{9} - \cdots$

$$\frac{9\tan^{-1}(x) - 9x + 3x^3}{5x^5} = \frac{9}{\underset{5\cdot 5}{25}} - \frac{9x^2}{\underset{5\cdot 7}{35}} + \frac{9x^4}{\underset{5\cdot 9}{45}} - \frac{9x^6}{5\cdot 11} + \frac{9x^8}{5\cdot 13} - \cdots$$

$$= \frac{9}{5}\left(\underset{k=0}{\frac{1}{5}} - \underset{k=1}{\frac{x^2}{7}} + \underset{k=2}{\frac{x^4}{9}} - \frac{x^6}{11} + \frac{x^8}{13} - \frac{x^{10}}{15} + \cdots\right)$$

in summation ?

Taylor series of $f(x)$ at $a=0$

So we can work w/ this near $x=0$ instead of original $f(x)$

$$= \frac{9}{5} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{5+2k}$$

what can we do with this?

$$\lim_{x \to 0} \frac{9\tan^{-1}(x) - 9(x) + 3x^3}{5x^5}$$ (usual way: l'Hospital's Rule)

x near 0
(a=0)
so we can
use the series
we found

$$= \lim_{x \to 0} \underbrace{\frac{9}{5}\left(\frac{1}{5} - \frac{x^2}{7} + \frac{x^4}{9} - \cdots\right)}_{\text{Taylor series}} = \frac{9}{25}$$

what about

$$\int \frac{9\tan^{-1}(x) - 9x + 3x^3}{5x^5}\,dx$$

$$= \int \frac{9}{5}\left(\frac{1}{5} - \frac{x^2}{7} + \frac{x^4}{9} - \frac{x^6}{11} + \frac{x^8}{13} - \cdots\right) dx$$

$$= \frac{9}{5}\left(\frac{x}{5} - \frac{x^3}{3\cdot 7} + \frac{x^5}{5\cdot 9} - \frac{x^7}{7\cdot 11} + \frac{x^9}{9\cdot 13} - \frac{x^{11}}{11\cdot 15} + \cdots\right) + C$$

1.5 $\quad k=0 \quad k=1 \quad k=2 \quad k=3$

$$= \boxed{\frac{9}{5}\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)(2k+5)} + C}$$

we can then do, for example

$$\int_0^1 \frac{9\tan^{-1}(x) - 9x + 3x^3}{5x^5}$$ estimate using 5^{th}-degree polynomial

$$= \frac{9}{5}\left(\underbrace{\frac{x}{5} - \frac{x^3}{3\cdot 7} + \frac{x^5}{5\cdot 9}}_{5^{th}\text{-degree}} - \frac{x^7}{7\cdot 11} + \frac{x^9}{9\cdot 13} + \cdots\right)\Big|_0^1$$

$$= \underbrace{\frac{9}{5}\left(\frac{1}{5} - \frac{1}{21} + \frac{1}{45}\right.}_{5^{th}\text{ degree estimate}} \left. - \frac{1}{77} + \frac{1}{117} + \cdots\right)$$

How accurate is the estimate?

≈ 0.3143

it's an alternating series, so the absolute of the first term we throw away is the max error

here, the max error is $\frac{9}{5}\left|-\frac{1}{77}\right| \approx 0.02338$

<u>example</u> Estimate $\int_0^{0.1} e^{-x^2}\,dx$ to within 10^{-8}

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \frac{(-x^2)^4}{4!} + \cdots$$

$$= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \cdots$$

$$\int_0^{0.1} e^{-x^2}\,dx = \int_0^{0.1}\left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \cdots\right)dx$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} - \frac{x^{11}}{11\cdot 5!} + \frac{x^{13}}{13\cdot 6!} - \cdots \Big|_0^{0.1}$$

$$= 0.1 - \underbrace{\frac{(0.1)^3}{3}}_{3\times10^{-4}} + \underbrace{\frac{(0.1)^5}{5\cdot 2!}}_{10^{-6}} - \underbrace{\frac{(0.1)^7}{7\cdot 3!}}_{2.38\times10^{-9}} + \frac{(0.1)^9}{9\cdot 4!} - \cdots$$

alternating, so magnitude of the first we throw away is max error

less than 10^{-8}, so stop adding up to the term before it

$$\approx 0.1 - \frac{(0.1)^3}{3} + \frac{(0.1)^5}{5\cdot 2!} \approx 0.099668$$

this is within 10^{-8} of true value of $\int_0^{0.1} e^{-x^2}\,dx$

we know the alternating Harmonic series converges

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots \quad \text{converges}$$

$$= S \quad \text{(it has a sum)}$$

what does it converge to? (exactly?)

from Table: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$

$$\ln(1+1) = \underbrace{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots}_{\text{Alt. Harmonic series}} = \ln(2)$$

So, the Alt. Harmonic series sums to $\ln(2)$ exactly.