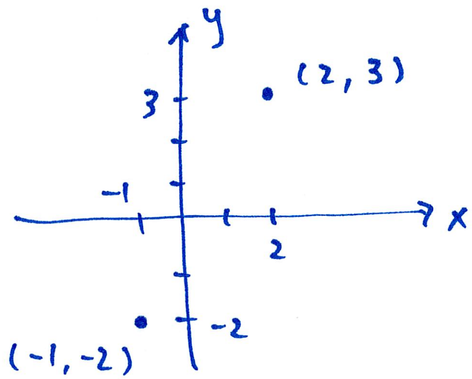
12.2 Polar Coordinates

in Rectangular / Cartesian coordinates, a point is located by (x, y)

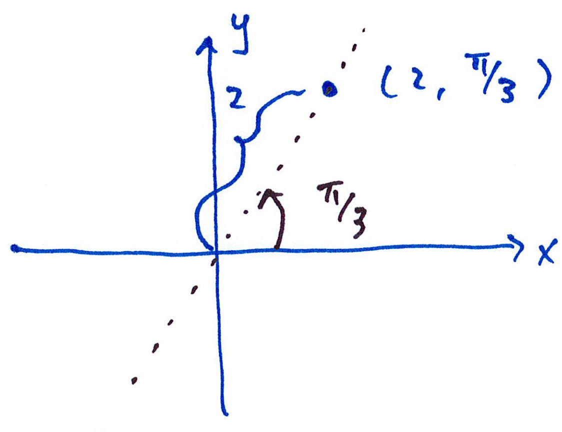


in Polar coordinates, a point is located as (r, θ)

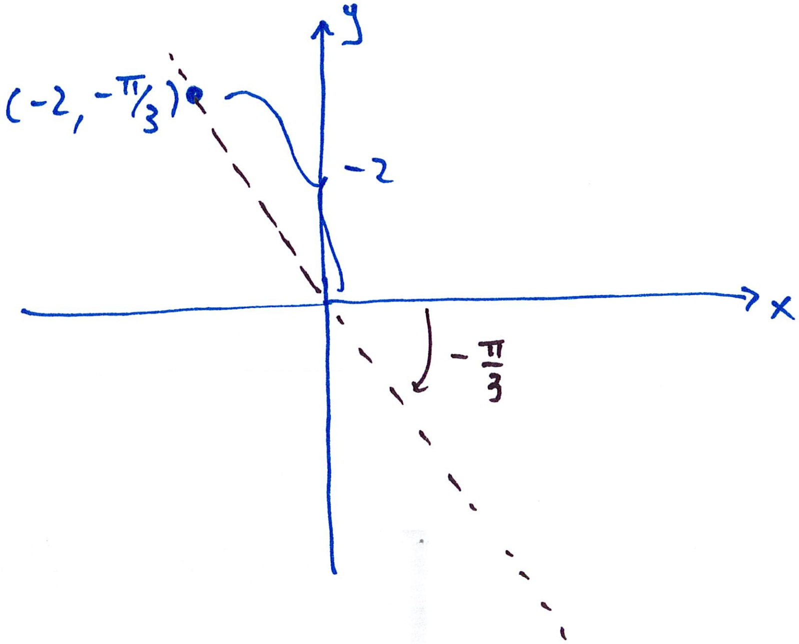
r: displacement from origin

θ: angle of a line through origin and the point in standard position (measured positive counterclockwise with respect x-axis)

for example, in Polar : $(2, \pi/3)$
$\downarrow$
θ



both r and θ can be negative

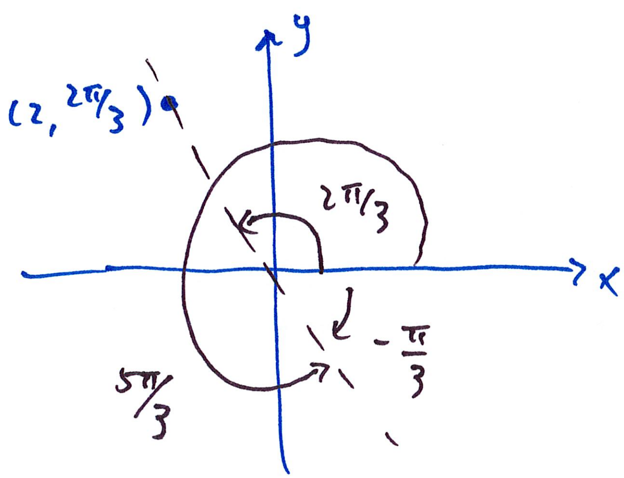


$\left(-2, -\frac{\pi}{3}\right)$

when $r > 0$, same quadrant as θ

when $r < 0$, opposite quadrant as θ

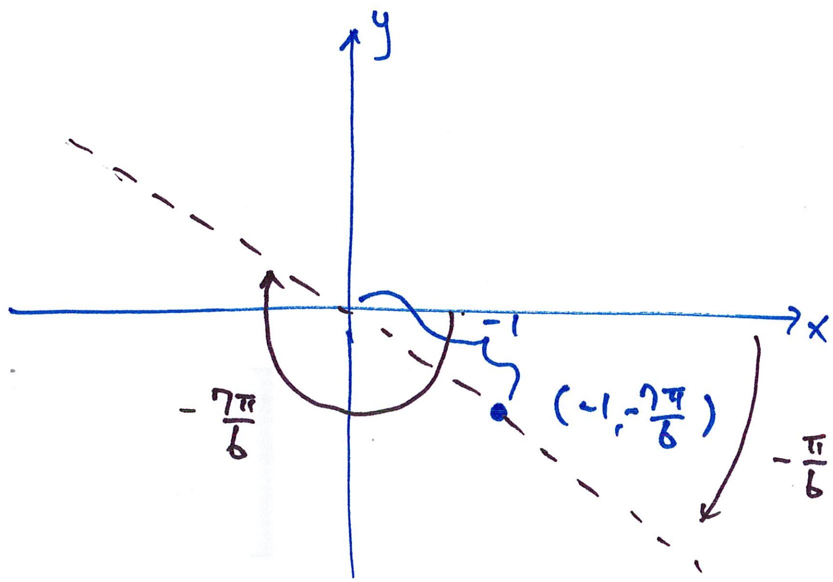
notice $(-2, -\pi/3)$ can also be expressed as $(2, 2\pi/3)$



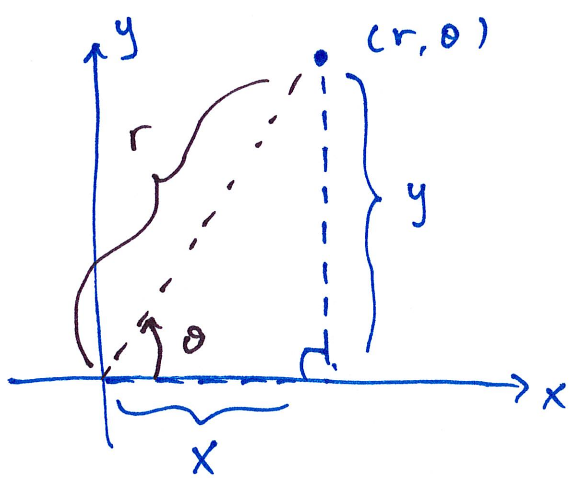
and also $\left(-2, \frac{5\pi}{3}\right)$

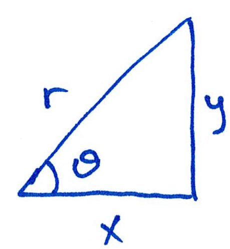
so, polar coordinate expression of a point is NOT unique point : $\left(-1, -\frac{7\pi}{6}\right)$

find two other ways to express this location : $\left(1, -\frac{\pi}{6}\right)$, $\cancel{\left(1, \frac{5\pi}{6}\right)}$, $\left(-1, \frac{5\pi}{6}\right)$



Conversion: Polar → Cartesian $(r, \theta) \to (x, y)$





$\cos\theta = \frac{x}{r} \to \boxed{x = r\cos\theta}$

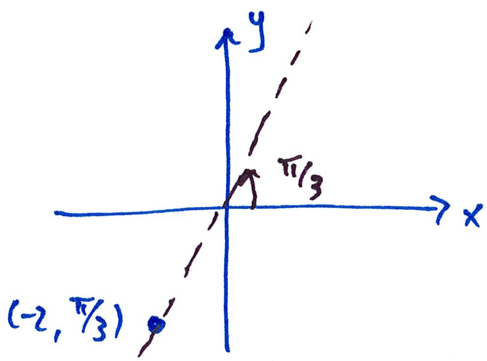
$\sin\theta = \frac{y}{r} \to \boxed{y = r\sin\theta}$

also, note $\boxed{x^2 + y^2 = r^2}$

these work in ALL quadrants
with pos/neg r, θ

example In polar: $(-2, \frac{\pi}{3})$

Cartesian?



in QIII: $x<0$, $y<0$

$$x = r\cos\theta = (-2)\cos\left(\frac{\pi}{3}\right) = -2\cdot\frac{1}{2} = -1$$

$$y = r\sin\theta = (-2)\sin\left(\frac{\pi}{3}\right) = -2\cdot\frac{\sqrt{3}}{2} = -\sqrt{3}$$

signs make sense

final check: is $x^2+y^2=r^2$?

$$(-1)^2+(-\sqrt{3})^2 = (-2)^2 \ ?$$

$$1+3=4 \qquad \text{yes.}$$

so in Cartesian, $\boxed{(-1, -\sqrt{3})}$

in Cartesian, the answer is unique

equations can be transformed, too

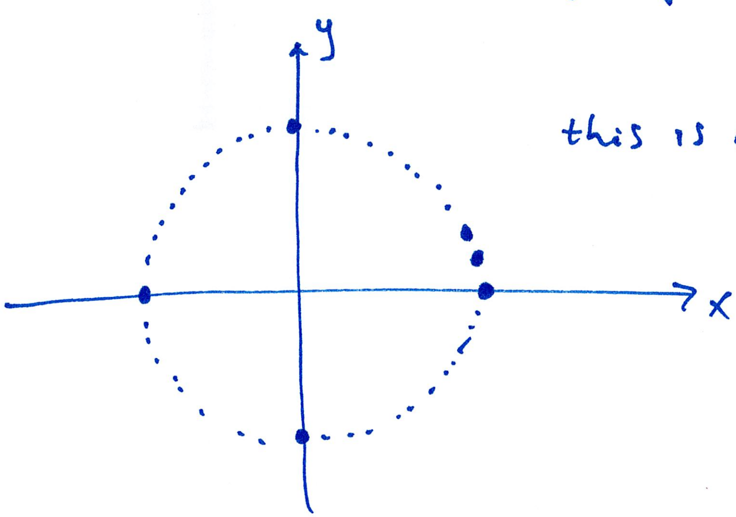
<u>example</u> $r = 3$

this is an equation in Polar, which is usually $r(\theta) = \dots$

(just like $y(x) = \dots$ in Cartesian)

in any variable is missing (in this case, θ) it means that variable can take on ANY value in its domain

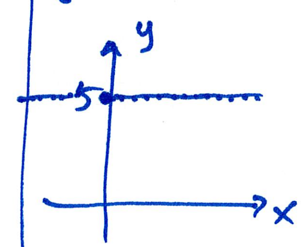
$r = 3$ θ can be anything



this is a circle radius 3

for example,

$y = 5$ (x is missing)



line

in Cartesian: $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$

$x^2 + y^2 = (3)^2$

$$\boxed{x^2 + y^2 = 9}$$

example

$$r = \frac{1}{2\cos\theta + 4\sin\theta}$$

$$r(2\cos\theta + 4\sin\theta) = 1$$

$$2\underbrace{\boxed{r\cos\theta}}_{x} + 4\underbrace{\boxed{r\sin\theta}}_{y} = 1$$

$$2x + 4y = 1$$

or $y = \frac{1}{4} - \frac{1}{2}x$ line

Conversion: Cartesian → Polar $(x,y) \to (r,\theta)$

we know: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$x^2+y^2=r^2 \longrightarrow r^2 = x^2+y^2$

or $\boxed{r = \sqrt{x^2+y^2} \quad \text{or} \quad r = -\sqrt{x^2+y^2}}$

we decide whether $r>0$ or $r<0$

$\dfrac{y}{x} = \dfrac{r\sin\theta}{r\cos\theta} = \tan\theta \to \boxed{\theta = \tan^{-1}\left(\dfrac{y}{x}\right)}$

make sure this agrees with the r we chose example

Cartesian : $(-1, -\sqrt{3})$

first, find r: $r^2 = x^2 + y^2 = (-1)^2 + (-\sqrt{3})^2 = 4$

so, $r = 2$ or $r = -2$

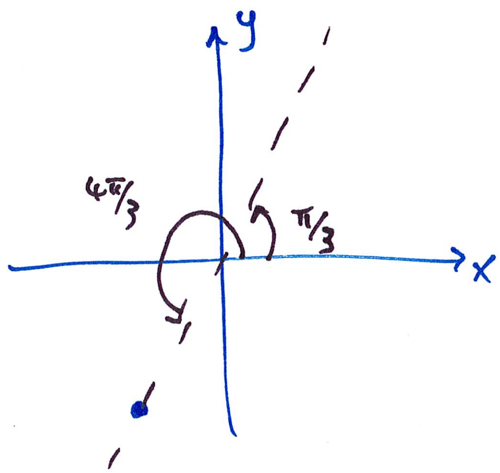
then θ: $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) = \tan^{-1}\left(\frac{-\sqrt{3}/2}{-1/2}\right) = \frac{4\pi}{3}$ (Q III)

$= \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right) = \frac{\pi}{3}$ (Q I)

this is in Q III

so, if we choose $r = 2$, we MUST go with $\theta = \frac{4\pi}{3}$ which is in Q III

if we choose $r = -2$, we MUST go with $\theta = \frac{\pi}{3}$ which is in Q I



so, in Polar, we have

$\left(2, \frac{4\pi}{3}\right)$ or $\left(-2, \frac{\pi}{3}\right)$

example What is $y = \frac{1}{x}$ in Polar?

again, we have the basic relationship

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ x^2+y^2 &= r^2 \end{aligned}}$$

$$y = \frac{1}{x}$$

$$r\sin\theta = \frac{1}{r\cos\theta}$$

$$r^2 = \frac{1}{\sin\theta\cos\theta} = \frac{1}{\sin\theta}\,\frac{1}{\cos\theta} = \csc\theta\sec\theta$$

$$\boxed{r^2 = \csc\theta\sec\theta}$$