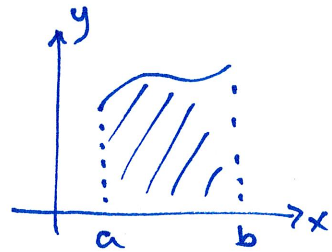
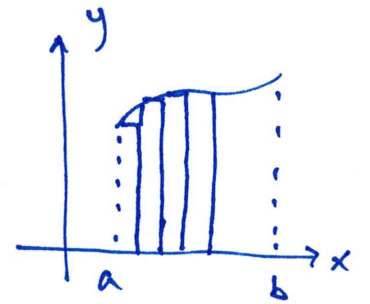
12.3 Areas and Lengths in Polar Coordinates

in Cartesian, area under $f(x)$ on $a \le x \le b$ is

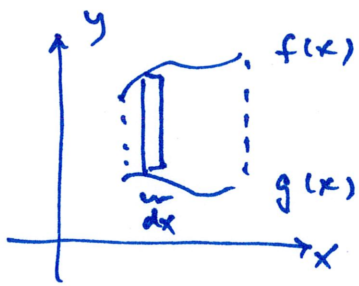
$$\int_a^b f(x)\,dx$$





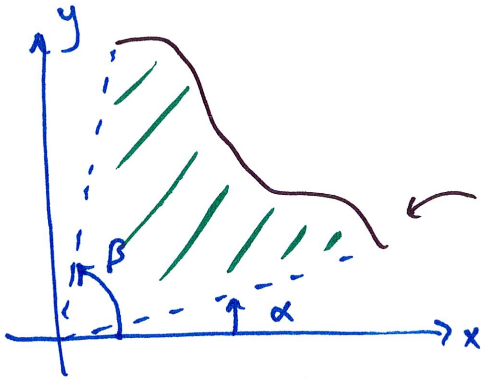
area = sum of infinitely-many thin rectangles

between curves



$$A = \int_a^b [f(x) - g(x)]\,dx$$

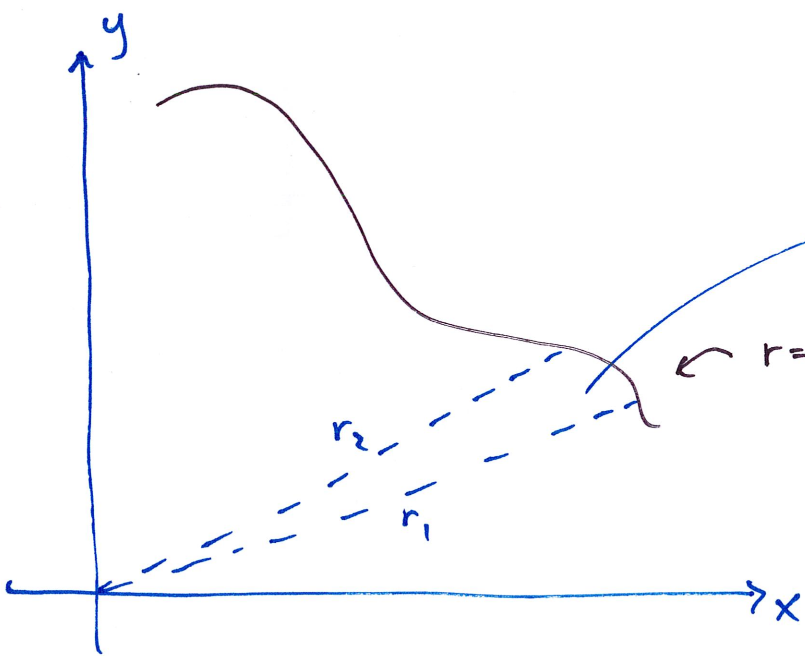
in polar, the idea is the same, but we sum infinitely-many thin circular segments



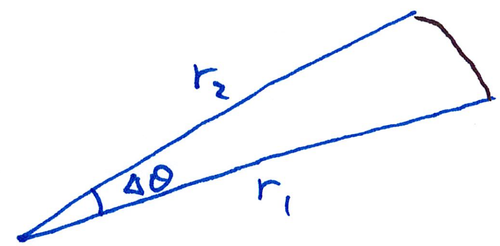
$r = f(\theta)$ (polar equation of curve)

area of the green region?

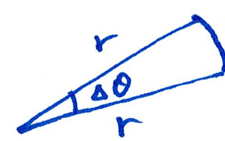
divide into thin circular segments



$r = f(\theta)$



since $\Delta\theta$ is small, $r_1 \approx r_2 = r$



circular arc = radius · angle

so, here, $r\Delta\theta$

also, the area of a circular segment is $\frac{1}{2}(\text{radius})^2(\text{angle})$

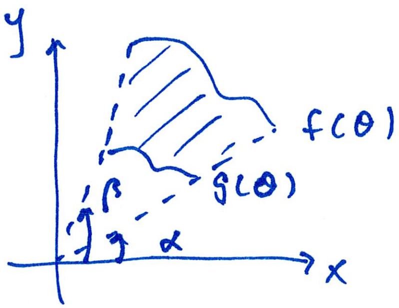
so, the thin segment has area of $\frac{1}{2}r^2\Delta\theta$

for the entire region, we sum infinitely-many of these and shrink $\Delta\theta \to d\theta$

so, area of a polar region is $\boxed{\int_\alpha^\beta \frac{1}{2}r^2\,d\theta}$

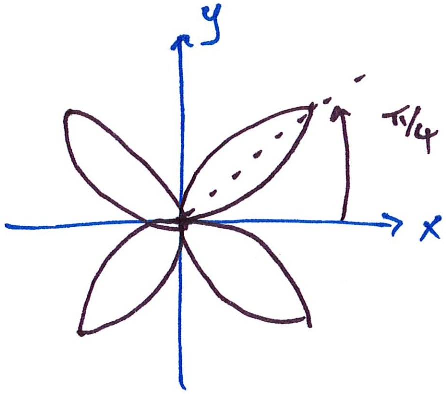
if $r=f(\theta)$, then area is also $\boxed{\int_\alpha^\beta \frac{1}{2}[f(\theta)]^2\,d\theta}$

region between curves:



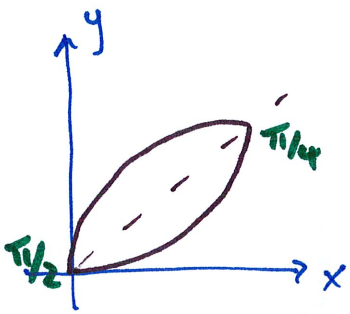
$$\boxed{\int_\alpha^\beta \left(\frac{1}{2}[f(\theta)]^2 - \frac{1}{2}[g(\theta)]^2\right) d\theta}$$

<u>example</u> Find the area of one petal of the rose $r = \sin 2\theta$



symmetric, so finding the area of one petal we can find area of any of the four.

for simplicity, let's find the QI petal



so, $\alpha = 0$, $\beta = \pi/2$

(or, do $\alpha = 0$, $\beta = \pi/4$, then double the result)

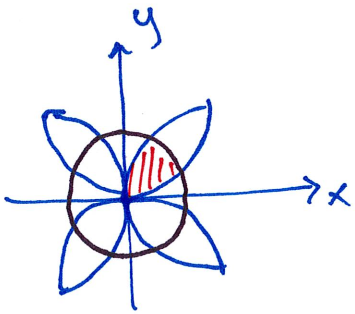
$$\int_0^{\pi/2} \frac{1}{2}[f(\theta)]^2 d\theta = \int_0^{\pi/2} \frac{1}{2}(\sin 2\theta)^2 d\theta$$

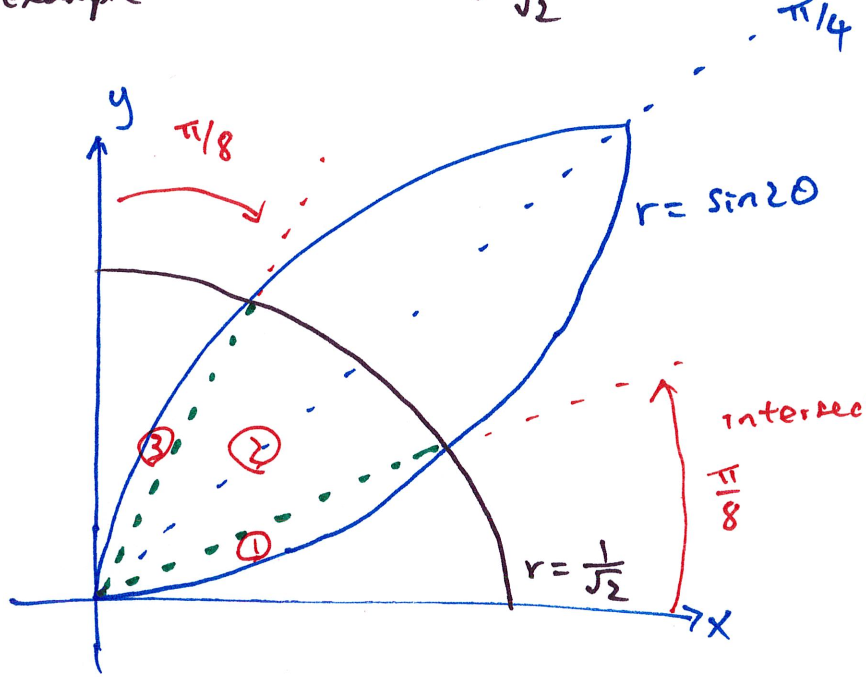
$$= \int_0^{\pi/2} \frac{1}{2}\sin^2 2\theta \, d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^2(2\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{1-\cos(4\theta)}{2} d\theta = \frac{1}{4}\int_0^{\pi/2} 1-\cos(4\theta)\, d\theta$$

$$= \frac{1}{4}\left(\theta - \frac{1}{4}\sin 4\theta\right)\Big|_0^{\pi/2} = \frac{1}{4}\left(\frac{\pi}{2} - 0\right) = \boxed{\frac{\pi}{8}}$$

example Area inside $\underbrace{r = \sin 2\theta}_{\text{rose in last example}}$ and $\underbrace{r = \frac{1}{\sqrt{2}}}_{\text{circle radius } \frac{1}{\sqrt{2}}}$ in Q I





intersection angle

$$\frac{1}{\sqrt{2}} = \sin 2\theta$$

$$2\theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

$$\theta = \frac{\pi}{8}$$

regions ① and ③ using $r = \sin 2\theta$ (① and ③ same area)

region ② using $r = \frac{1}{\sqrt{2}}$

how to calculate ① : $\displaystyle\int_0^{\frac{\pi}{8}} \frac{1}{2}(\sin 2\theta)^2 \, d\theta = \cdots = \frac{\pi}{32} - \frac{1}{16}$ (see last example)

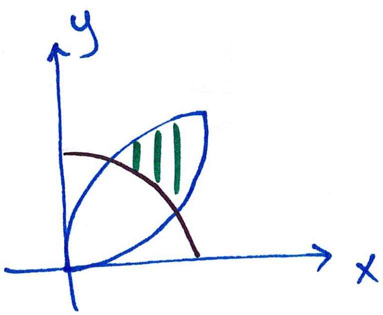
③ : same area : $\frac{\pi}{32} - \frac{1}{16}$

② : twice of area inside $r = \frac{1}{\sqrt{2}}$ from $\pi/8$ to $\pi/4$

$$2\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^2 d\theta = \frac{1}{2}\theta \Big|_{\frac{\pi}{8}}^{\frac{\pi}{4}} = \frac{\pi}{16}$$

total area : ① + ② + ③ $= \boxed{\frac{\pi}{8} - \frac{1}{8}}$

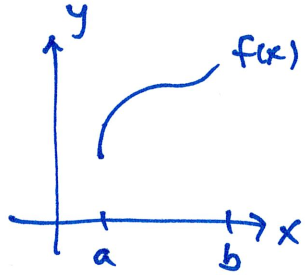
if we wanted the region outside circle $r = \frac{1}{\sqrt{2}}$ inside ~~rose~~ rose $r = \sin 2\theta$



we can do : $\underbrace{\text{rose petal area}}_{\text{1st example}} - \underbrace{\text{area inside}}_{\text{2nd example}}$

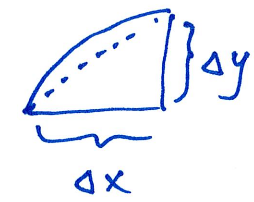
Length

in Cartesian

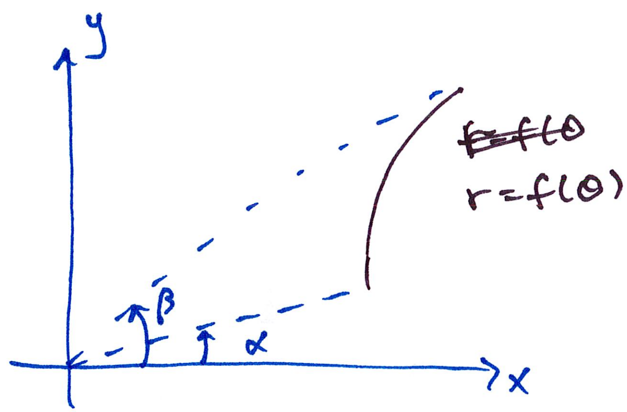


$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

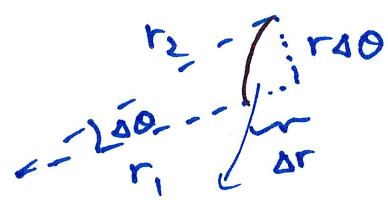
this came from $\sqrt{(\Delta x)^2 + (\Delta y)^2}$

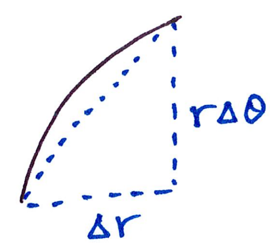


Polar: same idea, approximate curved section by straight line



take small slice:





the length is approximated by the hypotenuse

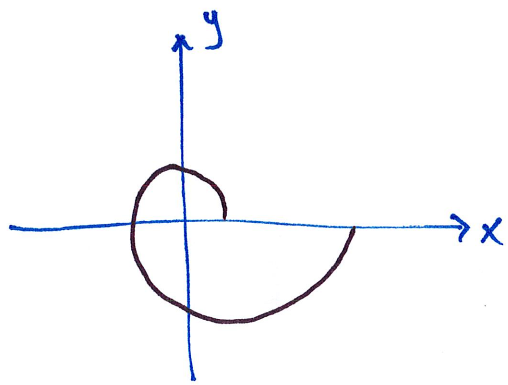
$$\sqrt{(\Delta r)^2 + (r\Delta\theta)^2} = \sqrt{(\Delta\theta)^2\left[\left(\frac{\Delta r}{\Delta\theta}\right)^2 + r^2\right]}$$

$$= \sqrt{\left(\frac{\Delta r}{\Delta\theta}\right)^2 + r^2}\,(\Delta\theta)$$

shrink $\Delta\theta \to d\theta$, so $\frac{\Delta r}{\Delta\theta} \to \frac{dr}{d\theta}$

then sum by integration : $\boxed{\displaystyle\int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta}$

<u>example</u> $r = e^\theta$ $\quad 0 \le \theta \le 2\pi$



"logarithmic spiral"

$$\text{length} = \int_0^{2\pi} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$$

$$r = e^\theta \quad \text{so} \quad \frac{dr}{d\theta} = e^\theta$$

$$\int_0^{2\pi} \sqrt{(e^\theta)^2 + (e^\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2e^{2\theta}}\, d\theta = \sqrt{2} \int_0^{2\pi} \sqrt{e^{2\theta}}\, d\theta$$

$$= \sqrt{2} \int_0^{2\pi} e^\theta\, d\theta = \sqrt{2}\, e^\theta \Big|_0^{2\pi} = \boxed{\sqrt{2}\,(e^{2\pi} - 1)}$$

Grade Cutoffs Estimate

- These are **not** finalized, just estimates

A+	95%
A	90%
A-	88%
B+	85%
B	80%
B-	78%
C+	72%
C	68%
C-	65%
D+	60%
D	55%