13.3 Dot Products

Scalar multiplication of a vector is simple : $k\langle a, b, c\rangle = \langle ka, kb, kc\rangle$

vector multiplication is more complicated : dot product (today)

cross ~~cp~~ product (next time)

Dot product :

$\vec{a} = \langle 1, 2, 3\rangle \quad \vec{b} = \langle 4, 5, 6\rangle$

$\vec{a}\cdot\vec{b} = 1\cdot 4 + 2\cdot 5 + 3\cdot 6 = 4+10+18 = 32$

dot product is a scalar

so, $\boxed{\langle a, b, c\rangle \cdot \langle e, f, g\rangle = ae + bf + cg}$

$\vec{a} = \langle 1, 2, 3\rangle$

$\vec{a}\cdot\vec{a} = \langle 1, 2, 3\rangle \cdot \langle 1, 2, 3\rangle = 1 + 4 + 9 = 14$

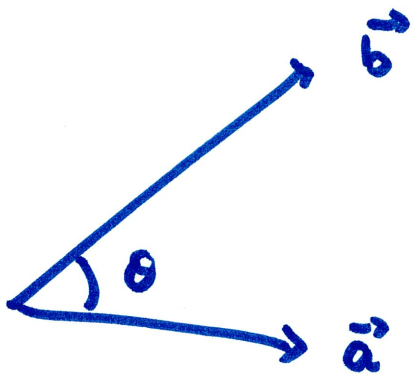
remember $|\vec{a}| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$

so we see $\boxed{\vec{a}\cdot\vec{a} = (|\vec{a}|)^2}$

$\vec{b}\cdot\vec{a} = 32$ so, $\boxed{\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}}$

another definition of dot product: $\boxed{\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta}$

θ : angle between $\vec{a}$ and $\vec{b}$



<u>example</u> Find angle between $\vec{a} = \langle 1, 2, -2 \rangle$ $\vec{b} = \langle 6, 0, -8 \rangle$

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

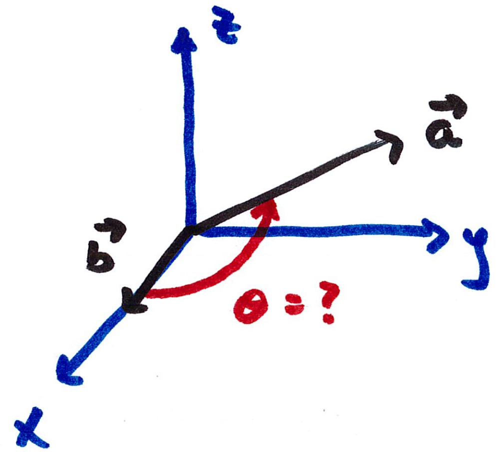
$$\vec{a}\cdot\vec{b} = \langle 1, 2, -2 \rangle \cdot \langle 6, 0, -8 \rangle = 6 + 0 + 16 = 22$$

$$|\vec{a}| = \sqrt{1+4+4} = 3 \qquad |\vec{b}| = \sqrt{36+64} = 10$$

$$22 = (3)(10)\cos\theta$$

$$\cos\theta = \frac{22}{30} \qquad \theta = \cos^{-1}\left(\frac{22}{30}\right) \approx 43^\circ$$

<u>example</u> what is the angle between $\vec{a} = 3\vec{i} + \vec{j} + 5\vec{k}$ and the x-axis?



we need ts a vector along the x-axis to be "$\vec{b}$"

any would do, since the magnitude does not affect θ

so, let's choose $\vec{b} = \langle 1, 0, 0 \rangle$

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$$

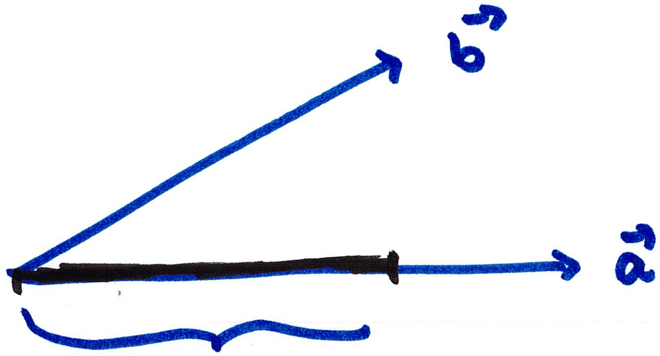
$$\vec{a} \cdot \vec{b} = \langle 3, 1, 5 \rangle \cdot \langle 1, 0, 0 \rangle = 3$$

$$|\vec{a}| = \sqrt{9+1+25} = \sqrt{35} \qquad |\vec{b}| = 1$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{3}{\sqrt{35}}$$ direction cosine

$\theta \approx 60°$ direction angle

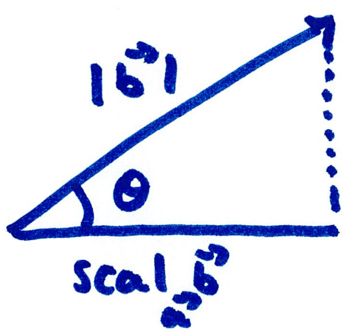
the dot product also allows us to find the <u>projection</u> of one vector onto another



the "shadow" of $\vec{b}$ onto $\vec{a}$

the length of this is called the <u>scalar projection</u> of $\vec{b}$ onto $\vec{a}$

written as $\text{scal}_{\vec{a}}\vec{b}$ (scalar projection of $\vec{a}$ onto $\vec{b}$ is $\text{scal}_{\vec{b}}\vec{a}$)



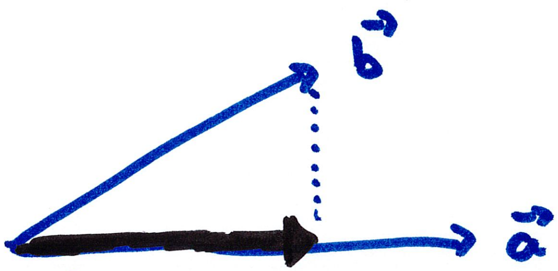
from trig. $\cos\theta = \dfrac{\text{scal}_{\vec{a}}\vec{b}}{|\vec{b}|}$

so, $\text{scal}_{\vec{a}}\vec{b} = |\vec{b}|\cos\theta$

from $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ we get $\cos\theta = \underbrace{\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}}_{\text{replace } \cos\theta \text{ in formula on last page}}$

now we get $\text{scal}_{\vec{a}}\vec{b} = \cancel{|\vec{b}|}\,\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cancel{|\vec{b}|}}$

or $\boxed{\text{scal}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$



↳ vector projection of $\vec{b}$ onto $\vec{a}$

it's a vector along $\vec{a}$ that has magnitude $\text{scal}_{\vec{a}}\vec{b}$

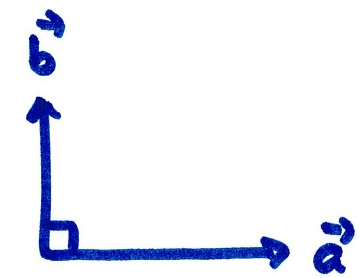
to give it a direction, we use a unit vector along $\vec{a}$: $\dfrac{\vec{a}}{|\vec{a}|}$

so, vector projection of $\vec{b}$ onto $\vec{a}$ is

$$\text{proj}_{\vec{a}}\vec{b} = \underbrace{\text{scal}_{\vec{a}}\vec{b}}_{\text{magnitude}} \ \underbrace{\frac{\vec{a}}{|\vec{a}|}}_{\text{direction}}$$

$$\boxed{\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\,\frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\,\vec{a}}$$

<u>example</u> $\vec{a} = \langle 2, -4, 2\rangle \quad \vec{b} = \langle 1, 5, 9\rangle$



$$\text{scal}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{2-20+18}{\sqrt{4+16+4}} = 0$$

$$\text{proj}_{\vec{a}}\vec{b} = \langle 0, 0, 0\rangle = \vec{0}$$

If $\vec{a}\cdot\vec{b} = 0$ (which results in $\text{scal}_{\vec{a}}\vec{b} = 0$)

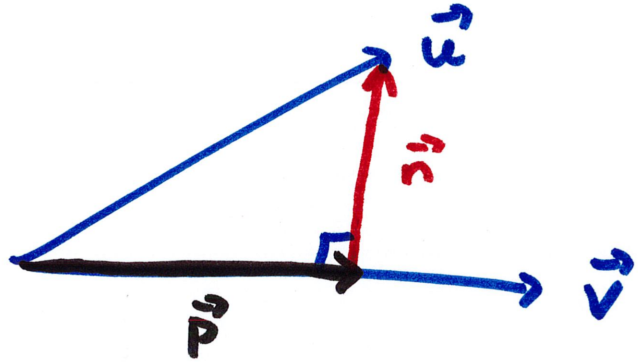
then $\vec{a} \perp \vec{b}$

($\vec{a}$ is orthogonal to $\vec{b}$)

<u>example</u> $\vec{u} = \langle -1, -4, -1 \rangle$ $\vec{v} = \langle 2, -2, 3 \rangle$

express $\vec{u}$ as $\vec{u} = \vec{p} + \vec{n}$

where $\vec{p}$ is parallel to $\vec{v}$ and $\vec{n}$ is orthogonal to $\vec{v}$



so we see $\vec{p} = \mathrm{proj}_{\vec{v}}\vec{u}$

$$= \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}$$

$$\vec{p} = \mathrm{proj}_{\vec{v}}\vec{u} = \frac{-2+8-3}{\left(\sqrt{4+4+9}\right)^2}\langle 2, -2, 3\rangle = \frac{3}{17}\langle 2, -2, 3\rangle = \left\langle \frac{6}{17}, -\frac{6}{17}, \frac{9}{17}\right\rangle$$

$\vec{u} = \vec{p} + \vec{n}$ so $\vec{n} = \vec{u} - \vec{p}$

$$= \langle -1, -4, -1\rangle - \left\langle \frac{6}{17}, \frac{-6}{17}, \frac{9}{17}\right\rangle$$

$$= \left\langle -\frac{23}{17}, -\frac{62}{17}, -\frac{26}{17}\right\rangle$$