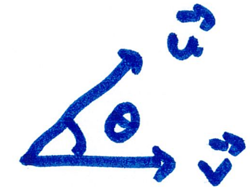
13.4 The Cross Product

another way vectors multiply

last time: dot product $\quad \vec{u}\cdot\vec{v} = |\vec{u}|\,|\vec{v}|\cos\theta$



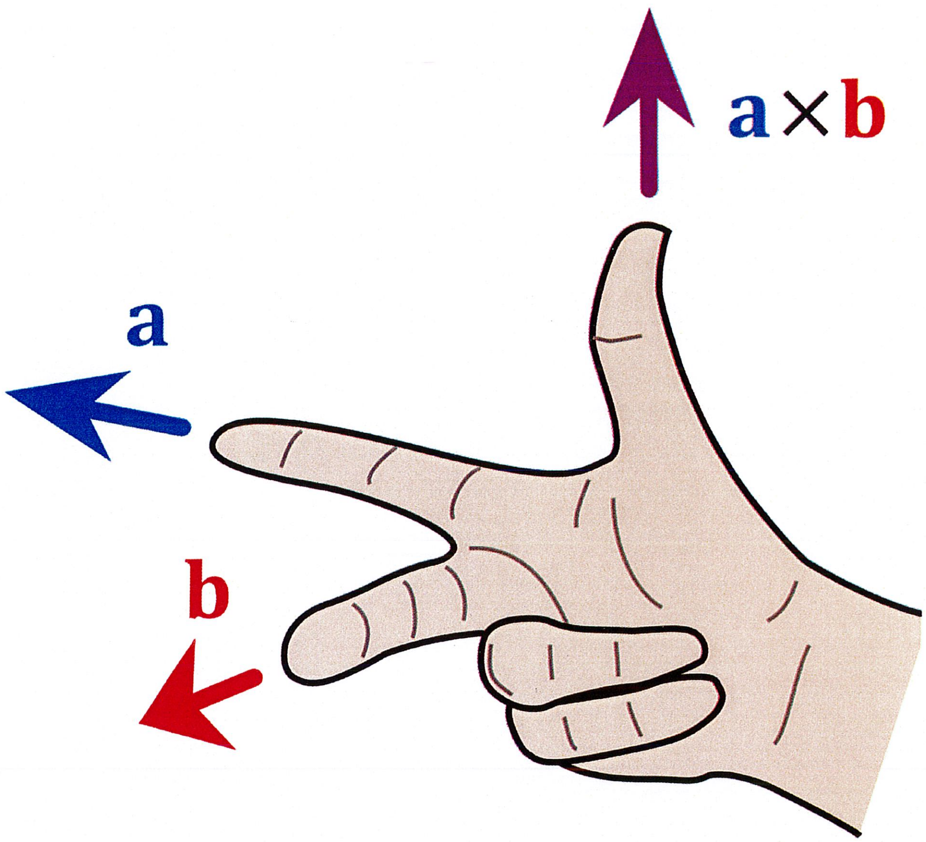
$\vec{u} = \langle a, b, c\rangle \quad \vec{v} = \langle d, e, f\rangle$

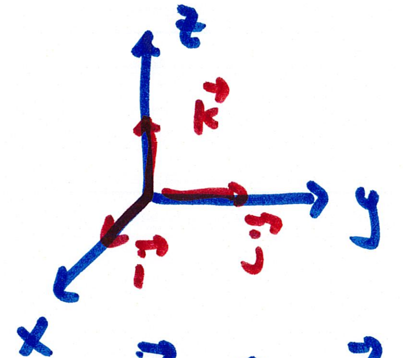
$\vec{u}\cdot\vec{v} = ad + be + cf \quad$ scalar

the cross product $\vec{u}\times\vec{v}$ is a vector

magnitude: $|\vec{u}\times\vec{v}| = |\vec{u}|\,|\vec{v}|\sin\theta$

direction: by the right-hand rule

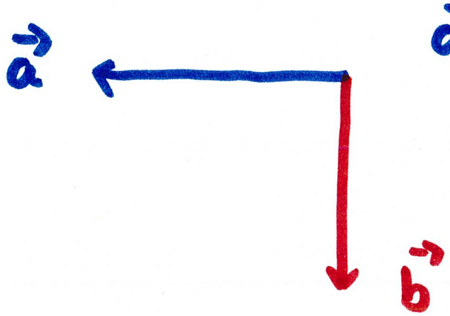




$\vec{i} \times \vec{j} = \vec{k}$

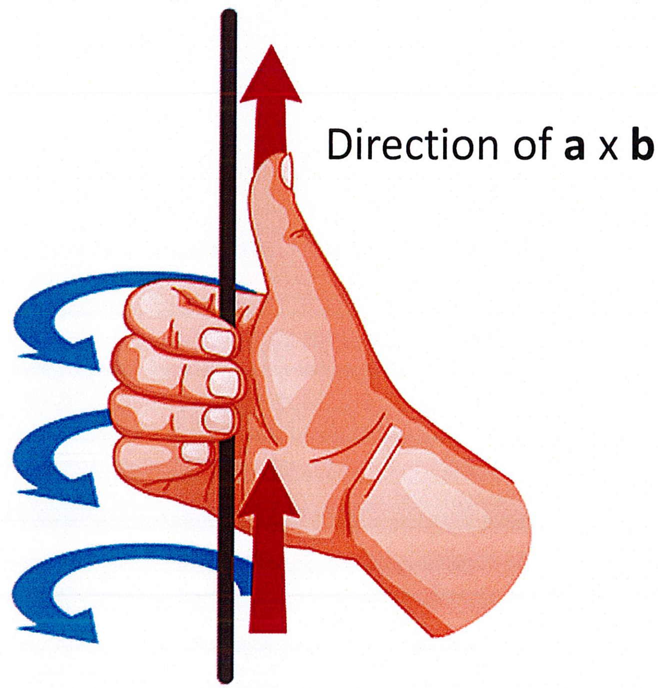
$\vec{j} \times \vec{i} = -\vec{k}$

CURL RIGHT HAND RULE



$\vec{a} \times \vec{b}$ points out of paper

$\vec{b} \times \vec{a}$ points into the paper



$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$ cross product: order matters

how to find cross product?

one way: given $\vec{u}$, $\vec{v}$, use $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$ to find the magnitude

then use right-hand rule for direction

another way: use determinant of a special matrix

2×2 matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ determinant is $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (3)(2)$
$= -2$

3×3 is more complicated but gives us the cross product

$\vec{u} = \langle 2, 1, 1 \rangle \qquad \vec{v} = \langle 5, 0, 1 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 5 & 0 & 1 \end{vmatrix}$$

first row: $\vec{i}, \vec{j}, \vec{k}$

second row: first vector in cross product

third row: second vector in cross product

reverse sign of $\vec{j}$

$$= \vec{i} \underbrace{\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}}_{} - \vec{j} \begin{vmatrix} 2 & 1 \\ 5 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & 1 \\ 5 & 0 \end{vmatrix}$$

determinant of matrix after blocking out the row and column $\vec{i}$ is in

$$= \vec{i}\,(1\cdot 1 - 0\cdot 1) - \vec{j}\,(2\cdot 1 - 5\cdot 1) + \vec{k}\,(2\cdot 0 - 5\cdot 1)$$

$$= \boxed{\vec{i} + 3\vec{j} - 5\vec{k} = \langle 1, 3, -5 \rangle} = \vec{u} \times \vec{v}$$

$\vec{u} = \langle 2, 1, 1 \rangle \quad \vec{v} = \langle 5, 0, 1 \rangle$

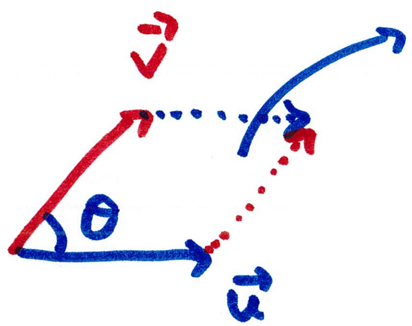
$\vec{u} \times \vec{v} = \langle 1, 3, -5 \rangle \qquad \vec{v} \times \vec{u} = \langle -1, -3, 5 \rangle$

notice: $(\vec{u} \times \vec{v}) \cdot \vec{u} = \langle 1, 3, -5 \rangle \cdot \langle 2, 1, 1 \rangle = 2 + 3 - 5 = 0$

$(\vec{u} \times \vec{v}) \cdot \vec{v} = \langle 1, 3, -5 \rangle \cdot \langle 5, 0, 1 \rangle = 5 + 0 - 5 = 0$

this means the cross product is orthogonal to both of its parent vectors.

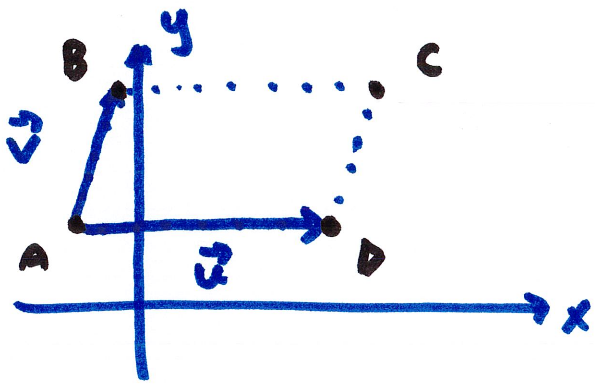
$|\vec{u} \times \vec{v}|$ also gives us the area of a parallelogram formed using $\vec{u}, \vec{v}$



area of this $= |\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$

example Find area of parallelogram with vertices

A $(-3, 4)$ B $(-1, 7)$ C $(3, 5)$ D $(1, 2)$



area $= |\vec{u} \times \vec{v}|$ or $|\vec{v} \times \vec{u}|$

$\vec{u} = \langle 4, -2 \rangle = \langle 4, -2, 0 \rangle$

$\vec{v} = \langle 2, 3 \rangle = \langle 2, 3, 0 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -2 & 0 \\ 2 & 3 & 0 \end{vmatrix}$$

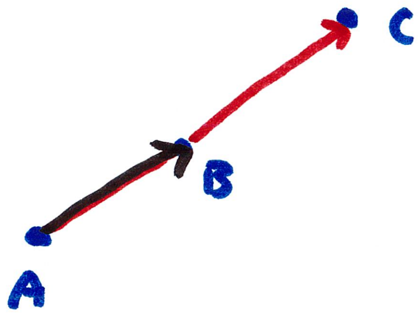
$$= \vec{i} \begin{vmatrix} -2 & 0 \\ 3 & 0 \end{vmatrix} - \vec{j} \begin{vmatrix} 4 & 0 \\ 2 & 0 \end{vmatrix} + \vec{k} \begin{vmatrix} 4 & -2 \\ 2 & 3 \end{vmatrix}$$

$$= \vec{i}(-2 \cdot 0 - 3 \cdot 0) - \vec{j}(4 \cdot 0 - 2 \cdot 0) + \vec{k}(4 \cdot 3 - 2 \cdot -2)$$

$$= 16 \vec{k}$$

$$|\vec{u} \times \vec{v}| = |16 \vec{k}| = \boxed{16}$$

collinear and coplanar vectors

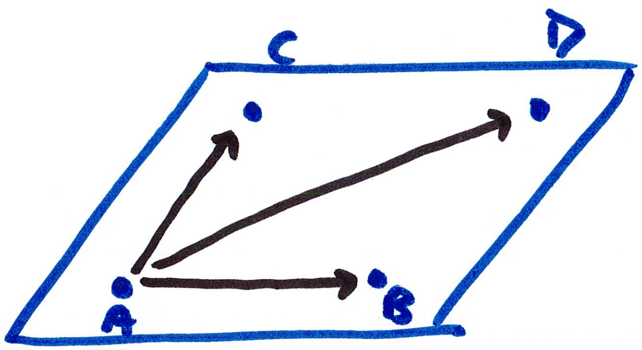


$\vec{AC}$ and $\vec{AB}$ are parallel
(A, B, C collinear)

note angle between $\vec{AC}$ and $\vec{AB}$ is 0

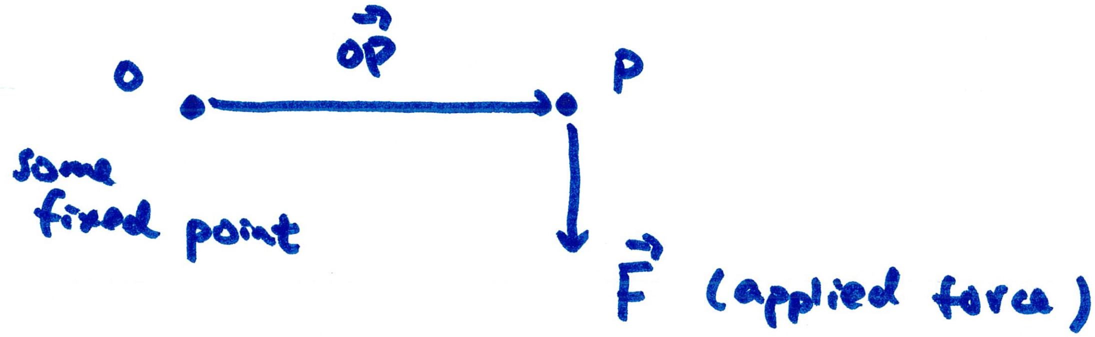
$$|\vec{AC} \times \vec{AB}| = |\vec{AC}|\,|\vec{AB}|\underbrace{\sin\theta}_{0} = 0$$

if A, B, C collinear then $|\vec{AC} \times AB|$ shown on left is 0



A, B, C, D coplanar (on same plane)

$\vec{AB} \times \vec{AC}$ and $\vec{AC} \times \vec{AB}$ are orthogonal to ANY vector in the plane in physics, torque / moment can be calculated using cross product



the torque about O is $|\vec{OP} \times \vec{F}|$

magnetism

magnetic field

$$\vec{F} = q(\vec{v} \times \vec{B})$$

charge

flow of current

Math Club callout

Wed. 8/31 Hicks G980D 7 pm.