6.2 Regions Between Curves

The area from $x=a$ to $x=b$ between x-axis and $y=f(x)$

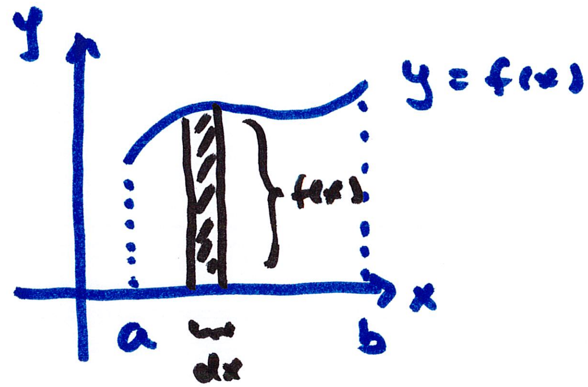
is $\int_a^b \underbrace{f(x)\,dx}$

area of rectangle

height $f(x)$ and width dx

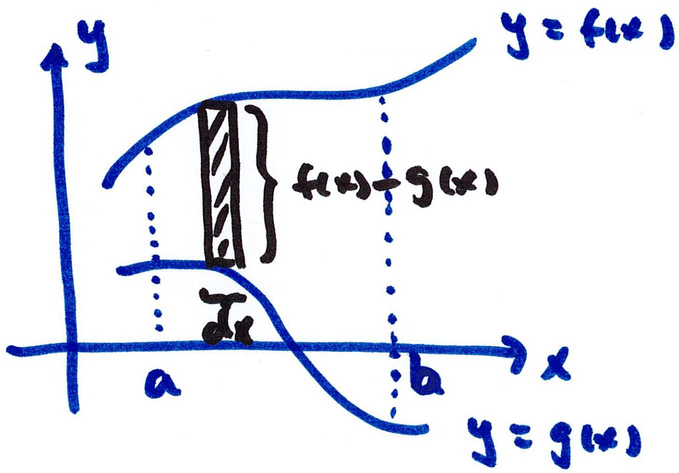
Sum up infinitely-many rectangles

from $x=a$ to $x=b$



now use the same idea to find area between two curves

$y=f(x)$, $y=g(x)$



each rectangle width dx

height $f(x) - g(x)$

now sum from $x=a$ to $x=b$

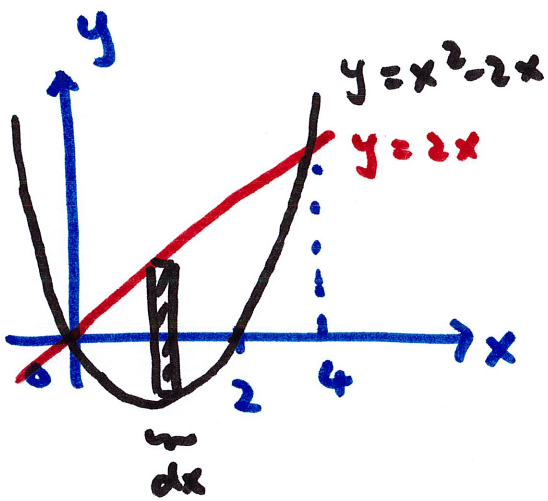
$$\boxed{\int_a^b [\underset{\text{top}}{f(x)} - \underset{\text{bottom}}{g(x)}]\,dx}$$

example Find the area of region bounded by

$$\underbrace{y = x^2 - 2x}_{\text{parabola opens up}} \quad \text{and} \quad \underbrace{y = 2x}_{\text{line}}$$

x-ints: $y = 0 = x^2 - 2x$

$= x(x-2) \rightarrow x = 0, \; x = 2$



start accumulating rectangles at $x=0$ and end at the intersection of $y = x^2 - 2x$ and $y = 2x$

$x^2 - 2x = 2x$

$x^2 - 4x = 0$

$x(x-4) = 0 \rightarrow x = 0, \; x = 4$

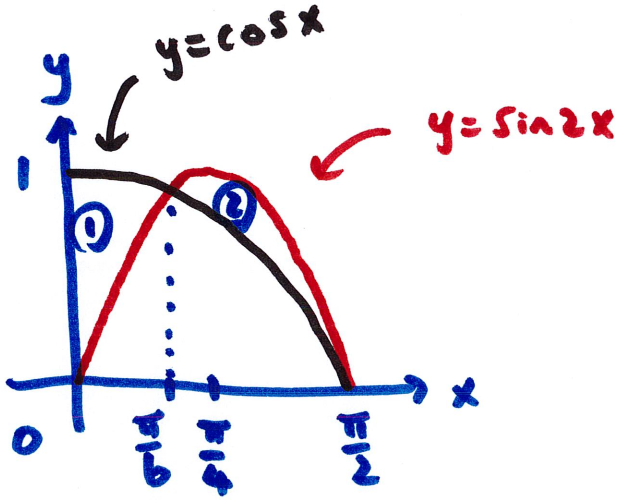
rectangle height: $(2x) - (x^2 - 2x) = 4x - x^2$

width: dx

$$\int_0^4 (4x - x^2)\,dx = \left(4 \cdot \frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^4 = 2x^2 - \frac{x^3}{3}\Big|_0^4$$

$$= \left(2(4)^2 - \frac{(4)^3}{3}\right) - \left(2(0)^2 - \frac{(0)^3}{3}\right) = 32 - \frac{64}{3} = \boxed{\frac{32}{3}}$$

__example__ Find area of region bounded by $y = \cos x$ and $y = \sin 2x$ from $x = 0$ to $x = \frac{\pi}{2}$



note on 1 $\cos x$ is above

on 2 $\sin 2x$ is above

intersection of $y = \cos x$ and $y = \sin 2x$

identity: $\sin 2x = 2 \sin x \cos x$

$\cos x = \sin 2x$

$\cos x = 2 \sin x \cos x$

$\cos x - 2 \sin x \cos x = 0$

$\cos x \, (1 - 2 \sin x) = 0$

$\cos x = 0$, $\quad 1 - 2 \sin x = 0$

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{2}$ $\quad x = \frac{\pi}{6}$ the intersection we want

$\int \cos x\,dx = \sin x + C$

$\int \sin x\,dx = -\cos x + C$

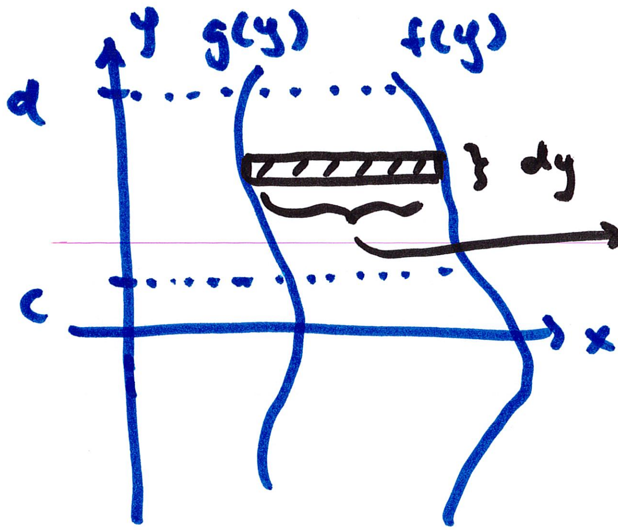
$$\underbrace{\int_0^{\frac{\pi}{6}} (\underset{\text{top}}{\cos x} - \underset{\text{bottom}}{\sin 2x})\,dx}_{1} + \underbrace{\int_{\pi/6}^{\pi/2} (\overset{\text{top}}{\sin 2x} - \overset{\text{bottom}}{\cos x})\,dx}_{2}$$

$$= \left(\sin x - -\frac{\cos 2x}{2}\right)\Bigg|_0^{\pi/6} + \left(-\frac{\cos 2x}{2} - \sin x\right)\Bigg|_{\pi/6}^{\pi/2}$$

$$= \left(\sin\frac{\pi}{6} + \frac{1}{2}\cos\frac{\pi}{3}\right) - \left(\sin 0 + \frac{1}{2}\cos 0\right) + \left(-\frac{1}{2}\cos\pi - \sin\frac{\pi}{2}\right) - \left(-\frac{1}{2}\cos\frac{\pi}{3} - \sin\frac{\pi}{6}\right)$$

$$= \left(\frac{1}{2} + \frac{1}{4}\right) - \left(0 + \frac{1}{2}\right) + \left(\frac{1}{2} - 1\right) - \left(-\frac{1}{4} - \frac{1}{2}\right) = \boxed{\frac{1}{2}}$$

Sometimes integrating in y is easier



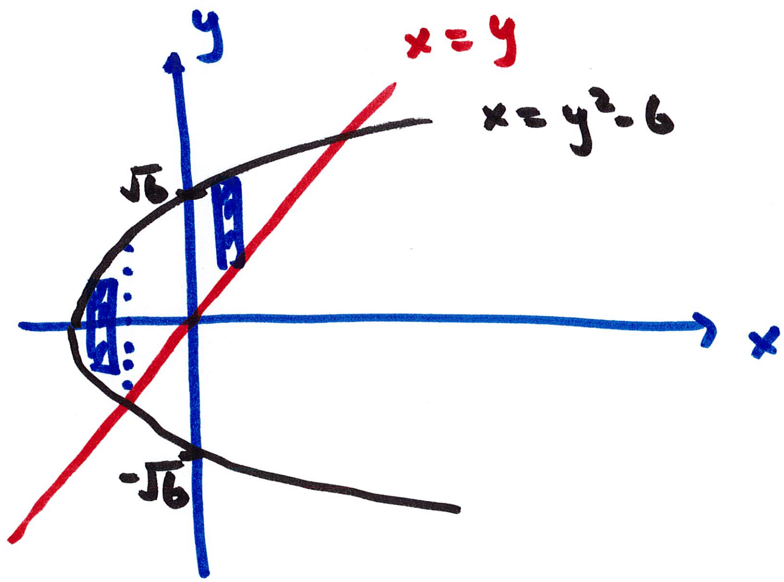
height : $f(y) - g(y)$

right left

area $= \int_c^d [f(y) - g(y)]\,dy$

right left

example Region bounded between $\underbrace{x = y^2 - 6}_{\text{parabola opening right}}$ $\overbrace{x = y}^{\text{line}}$

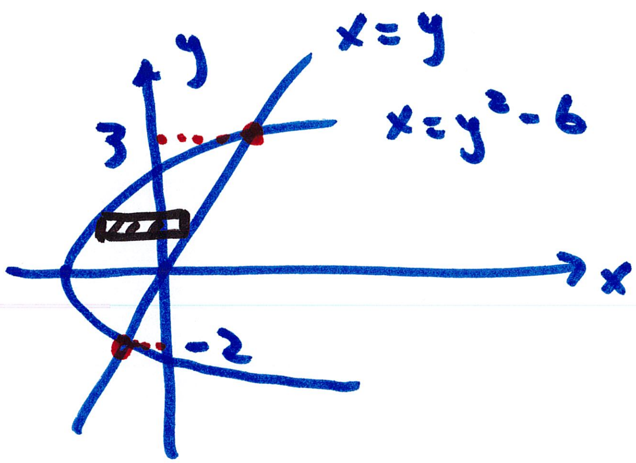


y-ints: $0 = y^2 - 6$

$y = \pm\sqrt{6}$

if we integrated in terms of x then rectangle is vertical

note left of dotted line, parabola is BOTH top and bottom but right of dotted line it's different → TWO INTEGRALS let's try integrating in y (horizontal rectangle)



intersection: $y = y^2 - 6$

$0 = y^2 - y - 6$

$= (y-3)(y+2)$

$y = -2, \quad y = 3$

rectangle height: $(y) - (y^2 - 6)$

right left

width: dy

$$\int_{-2}^{3} [y - (y^2 - 6)]\, dy = \cdots = \boxed{\frac{125}{6}}$$