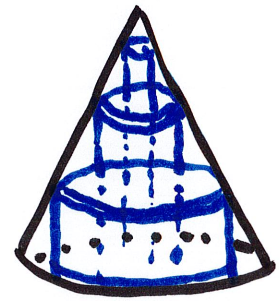
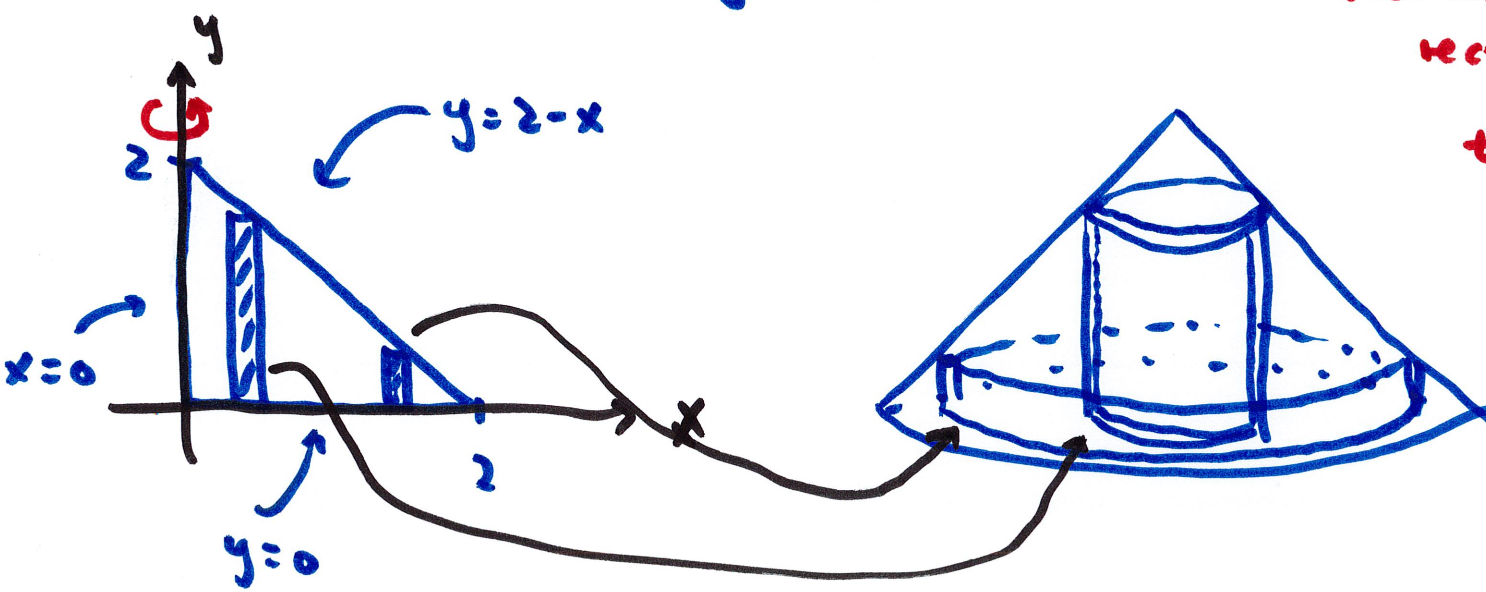
6.4 Volumes by Shells

last time: volumes by disks/washers

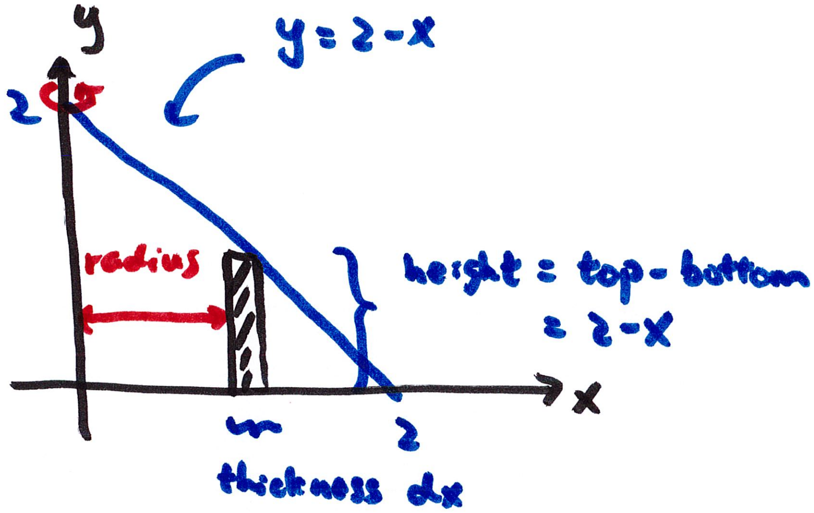
today: volumes by accumulating cylindrical shells



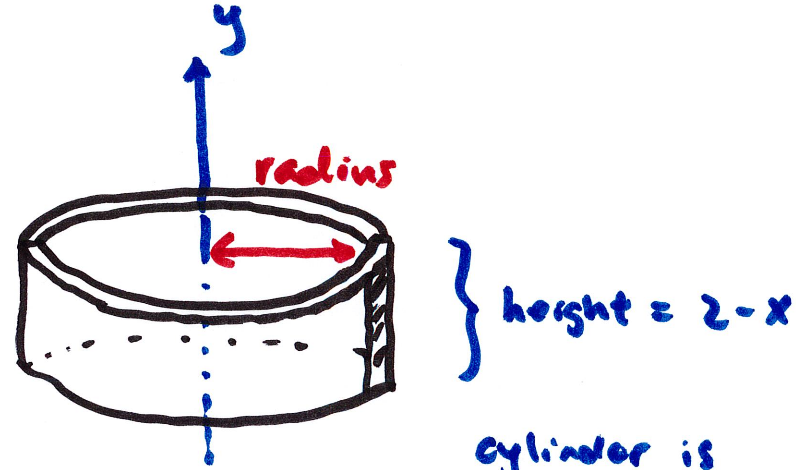
example Region bounded by $y = 2-x$, $y=0$, $x=0$
revolved around y-axis



shell method: set up rectangle parallel to axis of revolution
(disk: perpendicular)



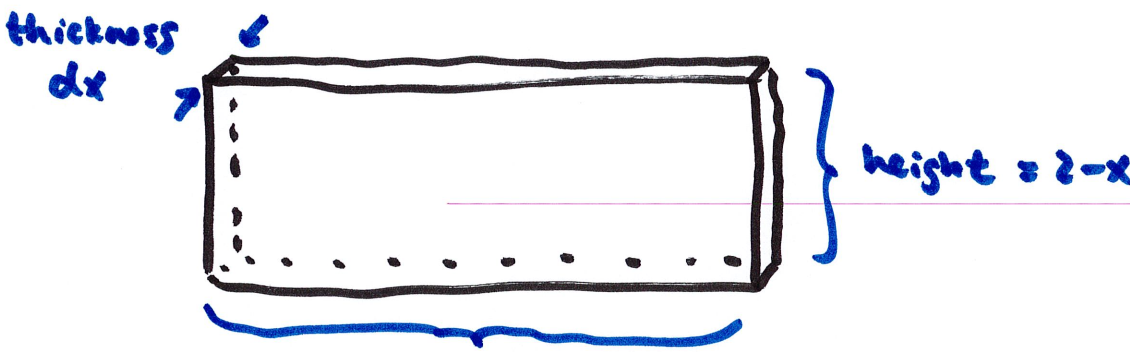
here, radius = x



cylinder is very thin so we only care about how far the edge is from axis of revolution (the radius)

volume of one cylinder?

cut cylinder, unwrap



length = circumference of cylindrical shell

$= 2\pi \cdot \text{radius} = 2\pi \cdot x$

volume of one slab:

length · height · thickness

$= (2\pi x)(2 - x)\, dx$

accumulate from $x = 0$ to $x = 2$ by integration

$x=2$, right end

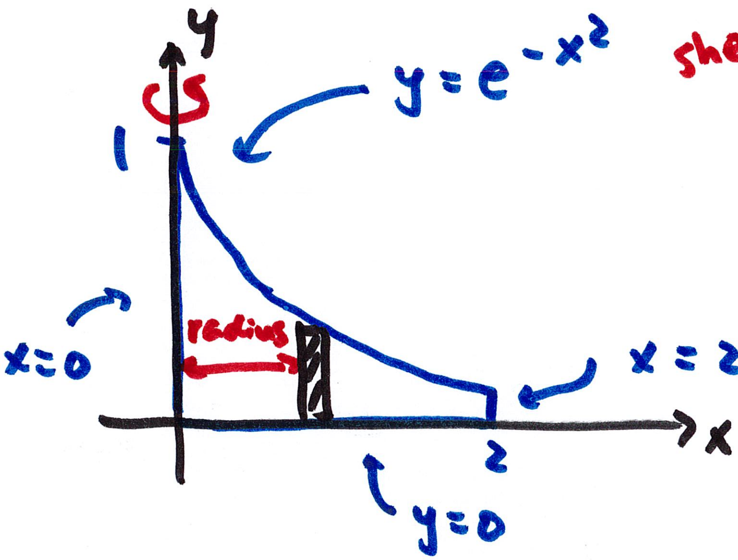
volume of whole cone:

$$V = \int_0^2 \underbrace{2\pi x\,(2-x)\,dx}_{\text{volume of one shell}} = \dots = \boxed{\frac{8\pi}{3}}$$

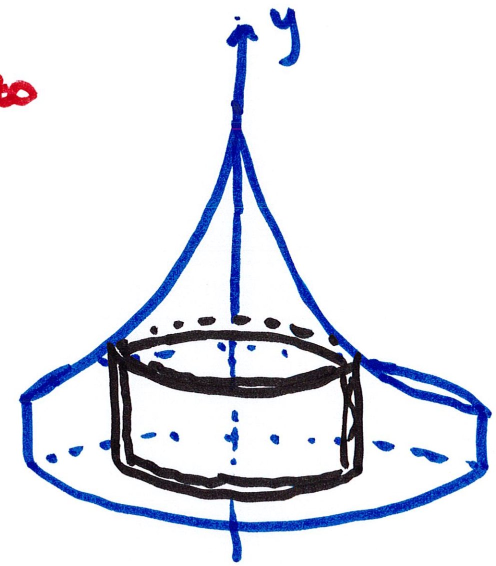
$x=0$, left end of region

<u>example</u> Region bounded by $y=e^{-x^2}$, $x=0$, $y=0$, $x=2$

revolve around y-axis



shell: rectangle parallel to axis



radius: from axis of revolution to rectangle

here, radius $= x$

height $= e^{-x^2}$

thickness $= dx$

volume of one shell $= 2\pi \cdot \text{radius} \cdot \text{height} \cdot \text{thickness}$

$= 2\pi \cdot x \cdot e^{-x^2}\,dx$

accumulate from left end of region to right end by integration

$$V = \int_0^2 2\pi x e^{-x^2} dx$$ by subs : $u = -x^2$, $du = -2x\,dx$

$$= \dots = \boxed{\pi(1-e^{-4})}$$

review u-sub

$$\int_0^2 2\pi x e^{-x^2} dx$$

$u = -x^2$

$du = -2x\,dx \rightarrow x\,dx = -\frac{1}{2}du$

w/ definite integral,
adjust bounds

$x = 2 \rightarrow u = -2^2 = -4$

$x = 0 \rightarrow u = -(0)^2 = 0$

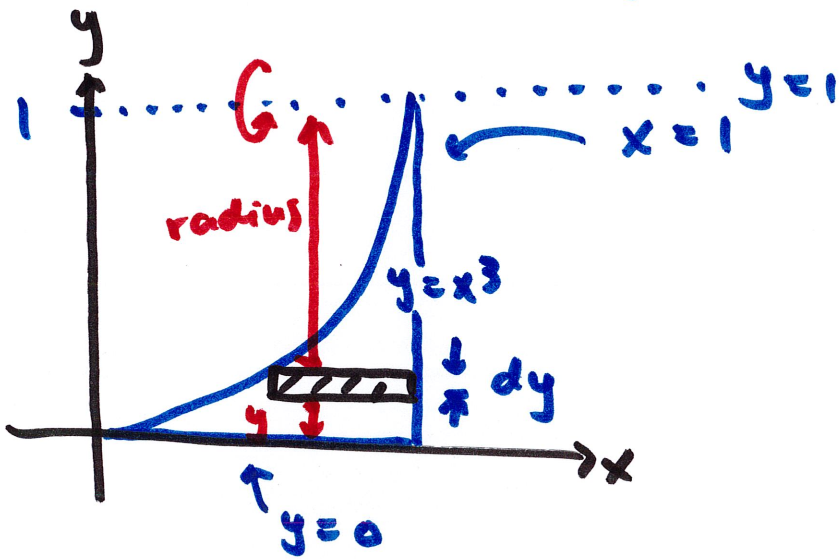
$$= \int_{u=0}^{u=-4} 2\pi \underbrace{e^{-x^2}}_{e^u} \underbrace{(x\,dx)}_{-\frac{1}{2}du}$$

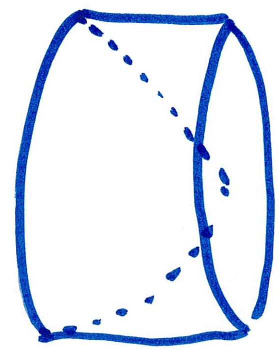
$$= \int_0^{-4} -\pi e^u du = -\pi \int_0^{-4} e^u du$$

$$= -\pi \left(e^u\right)\Big|_0^{-4} = -\pi\left(e^{-4} - e^0\right) = -\pi\left(e^{-4} - 1\right)$$

example Region bounded by $y = x^3$, $x = 1$, $y = 0$

revolved about $y = 1$





rectangle parallel to axis of revolution

"height" = right − left $= 1 - y^{1/3}$

↳ $y = x^3 \longleftrightarrow x = y^{1/3}$

radius = where is the rectangle measured from axis of revolution

$= 1 - y$ the rectangle is placed at distance of y from x-axis

volume of one shell $= 2\pi (1-y)(1-y^{1/3})\, dy$

accumulate all from $y=0$ to $y=1$

$$\int_0^1 2\pi(1-y)(1-y^{1/3})\,dy = \cdots = \boxed{\frac{5\pi}{14}}$$