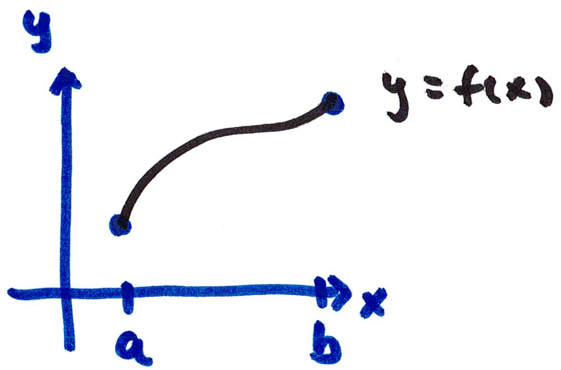
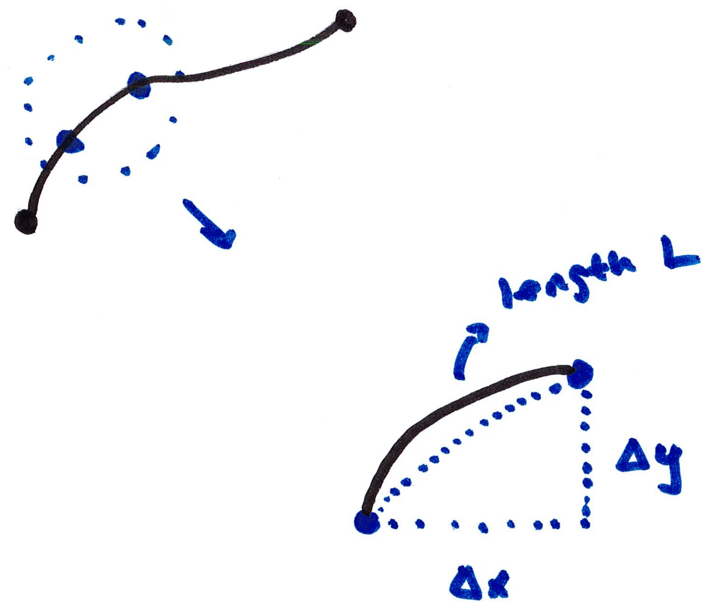
6.5 + 6.6 Length and Surface Area



How long is it from $x = a$ to $x = b$?

Cut into small pieces, find approximate length of each piece then accumulate by integration



if the points are close together

then $\sqrt{(\Delta x)^2 + (\Delta y)^2} \approx L$

$$L \approx \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(\Delta x)^2\left[1 + \frac{(\Delta y)^2}{(\Delta x)^2}\right]} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,(\Delta x)$$

now we shrink Δx : $\Delta x \to dx$

$$\frac{\Delta y}{\Delta x} \to \frac{dy}{dx}$$

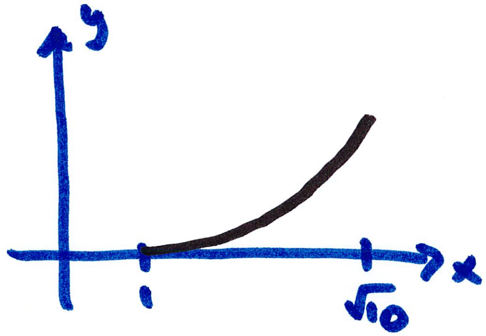
the length of segment then is $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

then accumulate all segments from $x=a$ to $x=b$

$$\boxed{\text{total length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx}$$

$y = f(x)$
must be continuous
on $a \le x \le b$
and $f'(x)$ must
exist

example $y = \frac{2}{3}(x^2-1)^{3/2}$ from $x=1$ to $x=\sqrt{10}$



$$L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

$$f'(x) = \frac{2}{3}\cdot\frac{3}{2}(x^2-1)^{1/2}\cdot 2x = 2x(x^2-1)^{1/2}$$

$$\int_1^{\sqrt{10}} \sqrt{1+[2x(x^2-1)^{1/2}]^2}\,dx = \int_1^{\sqrt{10}} \sqrt{1+4x^2(x^2-1)}\,dx$$

$$= \int_1^{\sqrt{10}} \sqrt{\underbrace{1+4x^4-4x^2}_{(2x^2-1)^2}}\,dx = \int_1^{\sqrt{10}} (2x^2-1)\,dx = \dots = \boxed{\frac{17}{3}\sqrt{10}+\frac{1}{3}}$$

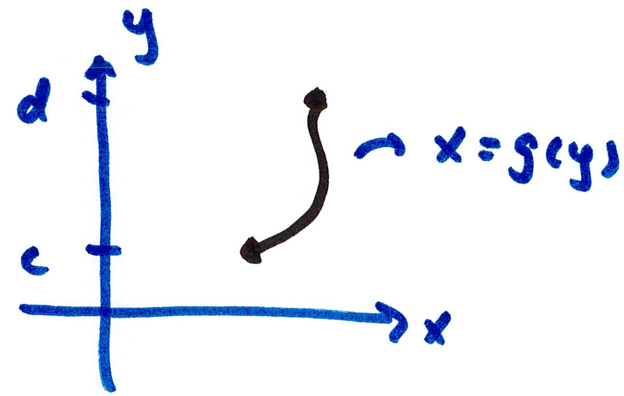
back to $\sqrt{(\Delta x)^2 + (\Delta y)^2}$

$$= \sqrt{(\Delta y)^2\left[\left(\frac{\Delta x}{\Delta y}\right)^2 + 1\right]} = \sqrt{1 + \left(\frac{\Delta x}{\Delta y}\right)^2}\,(\Delta y)$$

now $\Delta y \rightarrow dy \qquad \frac{\Delta x}{\Delta y} \rightarrow \frac{dx}{dy}$

so the equivalent length formula integrated with respect to y is

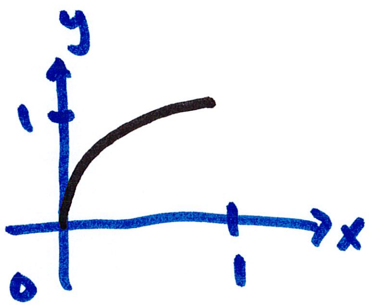
$$\int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$



$$= \int_c^d \sqrt{1 + (g'(y))^2}\, dy$$

Sometimes we switch the variable for ease of integration. Sometimes we <u>must</u> because, for example, $f'(x)$ does not exist somewhere on $a \le x \le b$

<u>example</u> $y = \sqrt[3]{x^2} = x^{2/3}$ from $x = 0$ to $x = 1$



$$\int_0^1 \sqrt{1 + (y')^2}\, dx$$

$$y' = \frac{2}{3} x^{-1/3} = \frac{2}{3x^{1/3}}$$ DNE at $x = 0$

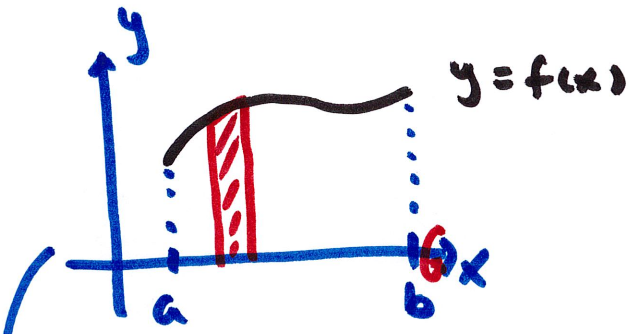
So must switch to integration in y: $\int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

$y = x^{2/3} \longrightarrow x = y^{3/2}$

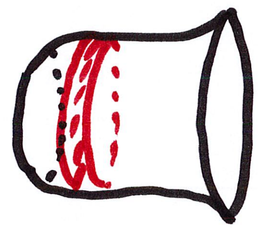
$\frac{dx}{dy} = \frac{3}{2} y^{1/2}$ exists on $0 \le y \le 1$ so ok to integrate

$$\int_0^1 \sqrt{1 + \left(\frac{3}{2} y^{1/2}\right)^2}\, dy = \int_0^1 \sqrt{1 + \frac{9}{4} y}\, dy = \dots = \text{something}$$

6.6 Surface area of solid of revolution



region bounded revolved around x-axis

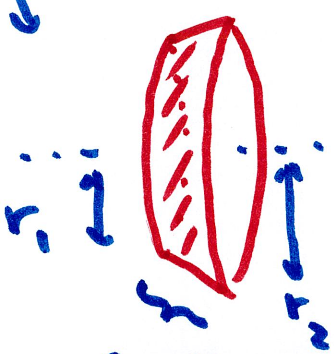


volume: disk or shell

surface area = ?

find area of one strip of the solid surface

accumulate by integration

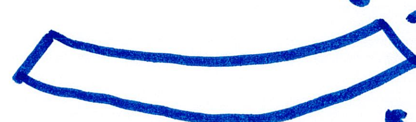


small piece
of the length
we ~~calculted~~
calculated earlier

cut, unwrap

circumference

d_1 (inner ~~radius~~)



} L

d_2 (~~outer~~) outer circumference

when the strip is very thin

$d_1 \approx d_2 = f(x) \cdot 2\pi$

} $\sqrt{1+[f'(x)]^2}\,dx$

$2\pi f(x)$

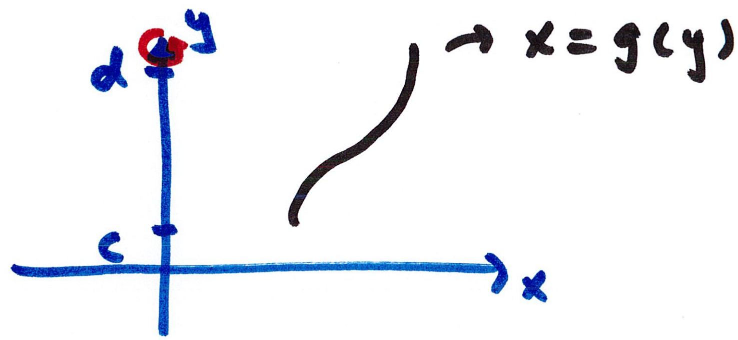
area of one strip: $2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$

accumulate from $x=a$ to $x=b$

total surface area = $\boxed{\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx}$

surface area of solid obtained by revolving about x-axis

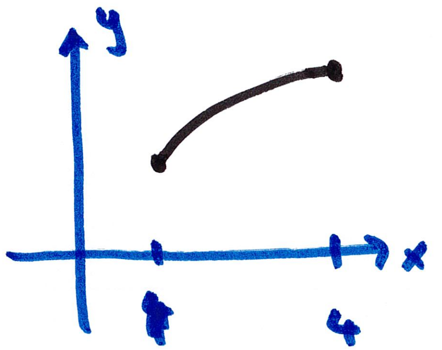
if around y-axis

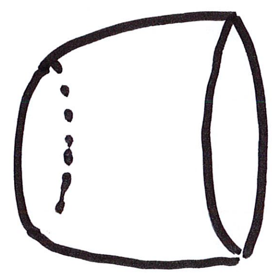


$$\int_c^d 2\pi g(y)\sqrt{1+[g'(y)]^2}\,dy$$

example Region bounded by $y=\sqrt{x}$, $x=1$, $x=4$, $y=0$

revolved around x-axis





Surface area = ?

$y = x^{1/2}$ $\quad y' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ is this defined on $x=1$ to $x=4$?

yes, ok to use the standard formula

$$\int_1^4 2\pi (x^{1/2}) \sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\, dx = 2\pi \int_1^4 \sqrt{x}\sqrt{1+\frac{1}{4x}}\, dx$$

$$= 2\pi \int_1^4 \sqrt{(x)\cdot\left(1+\frac{1}{4x}\right)}\, dx = 2\pi \int_1^4 \sqrt{x+\frac{1}{4}}\, dx$$

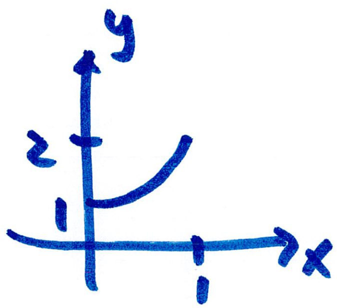
$u = x + \frac{1}{4} \quad du = dx \quad \ldots$

$= \ldots =$ something

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

this is actually a hard integral for some simple curves

for example, $y = x^2 + 1$ from $x = 0$ to $x = 1$



$$y' = 2x$$

$$L = \int_0^1 \sqrt{1 + (2x)^2}\, dx$$

$$= \int_0^1 \sqrt{1 + 4x^2}\, dx$$

this is not something we can do right now methods we can potentially use that we will learn in MA 162

later : trig substitution

integration by parts

power series expansion

for now, to solve this integral we can use an integral calculator

such as the one on Wolfram Alpha

www.wolframalpha.com/calculators/integral-calculator

or just google "wolfram integral calculator"

Please use it responsibly!

WolframAlpha computational intelligence.

$$\int_0^1 \sqrt{1+4x^2}\, d\,x$$

NATURAL LANGUAGE | MATH INPUT

BASIC MATH

$\frac{\square}{\square}$ | $\square^2$ | $\square^{\square}$ | $\sqrt{\square}$ | $\sqrt[3]{\square}$ | $\sqrt[\square]{\square}$ | ∞ | $-\infty$ | π | e | $e^{\square}$ | $\ln(\square)$ | $\log_{\square}(\square)$

$\log_{10}(\square)$ | $|\square|$ | $\square \le \square$ | $\square \ge \square$ | $\square \ne \square$

Definite integral

More digits | Step-by-step solution

$$\int_0^1 \sqrt{1+4x^2}\, dx = \frac{1}{4}\left(2\sqrt{5} + \sinh^{-1}(2)\right) \approx 1.4789$$

$\sinh^{-1}(x)$ is the inverse hyperbolic sine function

WolframAlpha computational intelligence.

$$\int_0^1 \sqrt{1+\frac{\pi^2}{4}\left(\sin\left(\frac{\pi x}{2}\right)\right)^2}\, d\,x$$

NATURAL LANGUAGE | MATH INPUT

BASIC MATH

∞ $-\infty$ π e $\ln(\square)$ $\log_{\square}(\square)$ $\log_{10}(\square)$ $|\square|$ $\square \le \square$ $\square \ge \square$ $\square \ne \square$

Definite integral

More digits

$$\int_0^1 \sqrt{1+\frac{\pi^2}{4}\sin^2\left(\frac{\pi x}{2}\right)}\, dx = \frac{2\,E\left(-\frac{\pi^2}{4}\right)}{\pi} \approx 1.4637$$

$E(m)$ is the complete elliptic integral of the second kind with parameter $m = k^2$