

8.4 Trigonometric Substitution (part 1)

$$\int \frac{1}{\sqrt{1-x^2}} dx$$

trig substitution: $x = \text{some trig function}$

let $x = \sin \theta \rightarrow$ why? we'll see in the following example

$$dx = \cos \theta d\theta$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \int \frac{1}{\sqrt{1-\sin^2 \theta}} \cos \theta d\theta = \int \frac{1}{\sqrt{\cos^2 \theta}} \cos \theta d\theta$$

$$= \int \frac{1}{\cos \theta} \cos \theta d\theta = \int d\theta = \theta + C \quad \text{now back to } x$$

$$x = \sin \theta \rightarrow \theta = \sin^{-1}(x)$$

$$= \boxed{\sin^{-1}(x) + C}$$

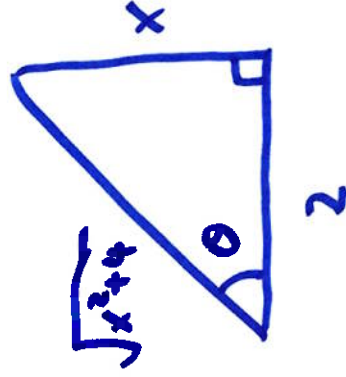
example

$$\int \frac{x^3}{\sqrt{x^2+4}} dx$$

to determine the right trig subs, look at the radical part

$$\sqrt{x^2+4}$$

set up a triangle with sides $\sqrt{x^2+4}$, x , 2
square root square root



hypotenuse: $\sqrt{x^2+4}$ because

$$(\sqrt{x^2+4})^2 = x^2 + 2^2$$

↖ ↗

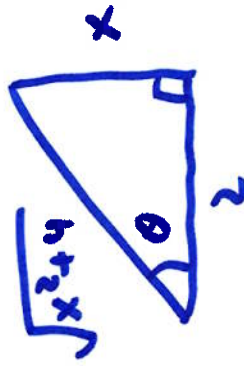
other two sides

two remaining sides: x , 2

rule of thumb: place constant as adjacent

relate x and θ in the simplest way using the triangle

↳ use the Simplest Sides



simplest way: $\tan = \frac{x}{2}$

$$x = 2 \tan \theta$$

$$dx = 2 \sec^2 \theta d\theta$$

$$\int \frac{x^3}{\sqrt{x^2 + 4}} dx = \int \frac{(2 \tan \theta)^3}{\sqrt{(2 \tan \theta)^2 + 4}} \cdot 2 \sec^2 \theta d\theta$$

$$= \int \frac{8 \tan^3 \theta}{\sqrt{4 \tan^2 \theta + 4}} \cdot 2 \sec^2 \theta d\theta \qquad \tan^2 \theta + 1 = \sec^2 \theta$$

$$\hookrightarrow \sqrt{4(\tan^2 \theta + 1)} = 2 \sqrt{\sec^2 \theta} = 2 \sec \theta$$

$$= \int \frac{8 \tan^3 \theta}{2 \sec \theta} \cdot 2 \sec^2 \theta d\theta = 8 \int \tan^3 \theta \sec \theta d\theta$$

$$= 8 \int \tan^2 \theta \sec \theta \tan \theta \, d\theta$$

$$u = \sec \theta$$

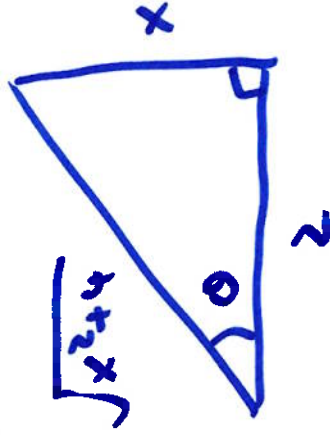
$$du = \sec \theta \tan \theta \, d\theta$$

$$= 8 \int (\sec^2 \theta - 1) \sec \theta \tan \theta \, d\theta$$

$$= 8 \int (u^2 - 1) du = 8 \left(\frac{u^3}{3} - u \right) + C = \frac{8}{3} \sec^3 \theta - 8 \sec \theta + C$$

not done yet, go back to x

bring triangle back:



$$\sec \theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{x^2 + 4}}{2}$$

$$= \frac{8}{3} \left(\frac{\sqrt{x^2 + 4}}{2} \right)^3 - 8 \left(\frac{\sqrt{x^2 + 4}}{2} \right) + C = \frac{1}{3} (x^2 + 4)^{3/2} - 4(x^2 + 4)^{1/2} + C$$

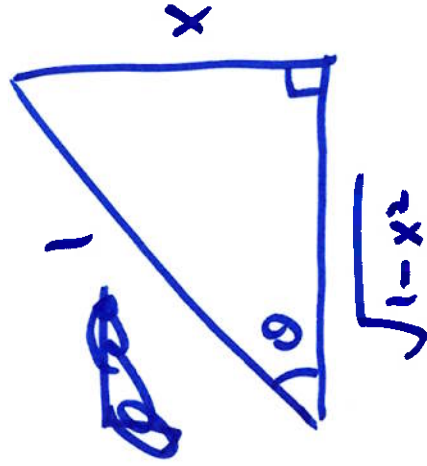
example

$$\int \frac{1}{(1-x^2)^{3/2}} dx = \int \frac{1}{(\sqrt{1-x^2})^3} dx$$

construct triangle w/ sides: $\sqrt{1-x^2}$, 1 , x



determine hypotenuse: 1 , because $1^2 = (\sqrt{1-x^2})^2 + x^2$



remaining sides: $\sqrt{1-x^2}$, x

neither is constant, so we take
the one containing a constant
as adjacent

relate x and θ using simplest sides: $\sin \theta = \frac{x}{1}$

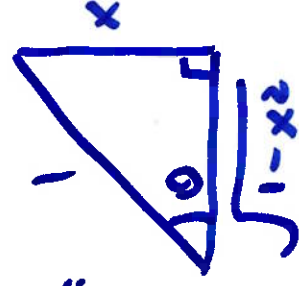
$$x = \sin \theta$$

$$dx = \cos \theta d\theta$$

$$\int \frac{1}{(1-x^2)^{3/2}} dx = \int \frac{1}{(1-\sin^2 \theta)^{3/2}} \cos \theta d\theta$$

$$= \int \frac{1}{(\cos^2 \theta)^{3/2}} \cos \theta d\theta = \int \frac{1}{\cos^3 \theta} \cos \theta d\theta = \int \frac{1}{\cos^2 \theta} d\theta$$

$$= \int \sec^2 \theta d\theta = \tan \theta + C \quad \text{go back to } x :$$



$$= \boxed{\frac{x}{\sqrt{1-x^2}} + C}$$

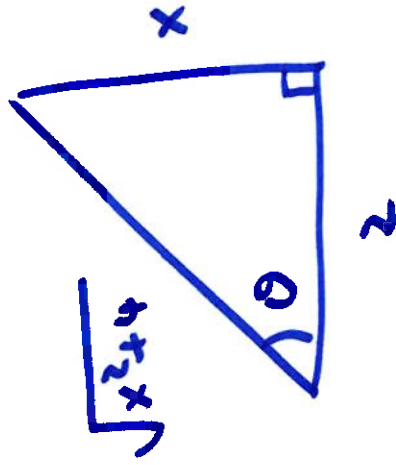
$$\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{x}{\sqrt{1-x^2}}$$

Example

$$\int_0^2 \frac{x^2}{x^2+4} dx = \int_0^2 \frac{x^2}{(\sqrt{x^2+4})^2} dx$$

triangle w/ sides: $\sqrt{x^2+4}$, x , 2

hypotenuse: $\sqrt{x^2+4}$ adjacent: 2



$$\tan \theta = \frac{x}{2}$$

$$x = 2 \tan \theta$$

$$dx = 2 \sec^2 \theta d\theta$$

$$\theta = \tan^{-1} \left(\frac{x}{2} \right)$$

now change integration limits to refer to θ

upper limit: $x=2 \rightarrow \theta = \tan^{-1} \left(\frac{2}{2} \right) = \tan^{-1}(1) = \frac{\pi}{4}$

lower limit: $x=0 \rightarrow \theta = \tan^{-1} \left(\frac{0}{2} \right) = \tan^{-1}(0) = 0$

$$\int_0^2 \frac{x^2}{x^2+4} dx = \int_0^{\pi/4} \frac{(2 \tan \theta)^2}{(2 \tan \theta)^2 + 4} \cdot 2 \sec^2 \theta d\theta$$

$$= \int_0^{\pi/4} \frac{4 \tan^2 \theta}{4(\tan^2 \theta + 1)} \cdot 2 \sec^2 \theta d\theta = \int_0^{\pi/4} \frac{\cancel{4} \tan^2 \theta}{\cancel{4} \sec^2 \theta} \cdot 2 \sec^2 \theta d\theta$$

$$= 2 \int_0^{\pi/4} \tan^2 \theta d\theta = 2 \int_0^{\pi/4} (\sec^2 \theta - 1) d\theta$$

$$= 2 \left(\int_0^{\pi/4} \sec^2 \theta d\theta - \int_0^{\pi/4} d\theta \right)$$

$$= 2 \left(\tan \theta \Big|_0^{\pi/4} - \theta \Big|_0^{\pi/4} \right) = 2 \left(1 - 0 - \frac{\pi}{4} - 0 \right) = \boxed{2 \left(1 - \frac{\pi}{4} \right)}$$