13.3 Dot Product

multiply scalars: $3 \cdot 5 = 15 \rightarrow a \cdot b = ab$

multiply vectors are more complicated: dot product (today)

cross product (next time)

Dot product: if $\vec{u} = \langle a, b \rangle$ $\vec{v} = \langle c, d \rangle$

then dot product of $\vec{u}$ and $\vec{v}$ is $\vec{u} \cdot \vec{v} = ac + bd$

example: $\vec{u} = \langle 1, 2 \rangle$ $\vec{v} = \langle 3, 4 \rangle$

$\vec{u} \cdot \vec{v} = 1 \cdot 3 + 2 \cdot 4 = 11$

example: $\vec{u} = \langle 1, 2, 3 \rangle$ $\vec{v} = \langle 4, 5, 7 \rangle$

$\vec{u} \cdot \vec{v} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 7 = 35$

note the dot product is a scalar

$\vec{u} = \langle 1, 2 \rangle$

$\vec{u} \cdot \vec{u} = \langle 1, 2 \rangle \cdot \langle 1, 2 \rangle = 1^2 + 2^2$

recall $|\vec{u}| = |\langle 1, 2 \rangle|$

$= \sqrt{1^2 + 2^2}$

$= \left(|\vec{u}|\right)^2$

square of magnitude

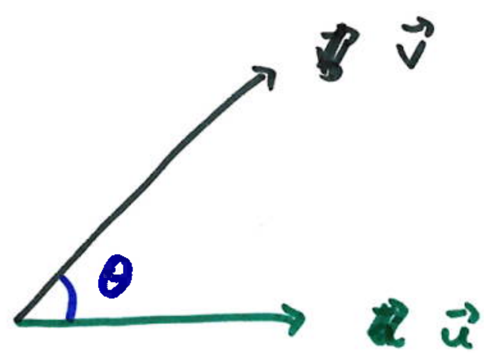
$\vec{u} = \langle 1, 2 \rangle \qquad \vec{v} = \langle 3, 4 \rangle$

$\vec{v} \cdot \vec{u} = 1 \cdot 3 + 2 \cdot 4 = 11 = \vec{u} \cdot \vec{v}$

$\boxed{\text{in general, } \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}}$

order doesn't matter

the geometric definition of dot product



θ: included angle (angle between $\vec{u}$, $\vec{v}$)

$$\boxed{\vec{u}\cdot\vec{v} = |\vec{u}|\,|\vec{v}|\cos\theta}$$

<u>example</u> Find the angle between $\vec{u} = \langle 1, 2, -2 \rangle$, $\vec{v} = \langle 6, 0, -8 \rangle$

$\vec{u}\cdot\vec{v} = |\vec{u}|\,|\vec{v}|\cos\theta$

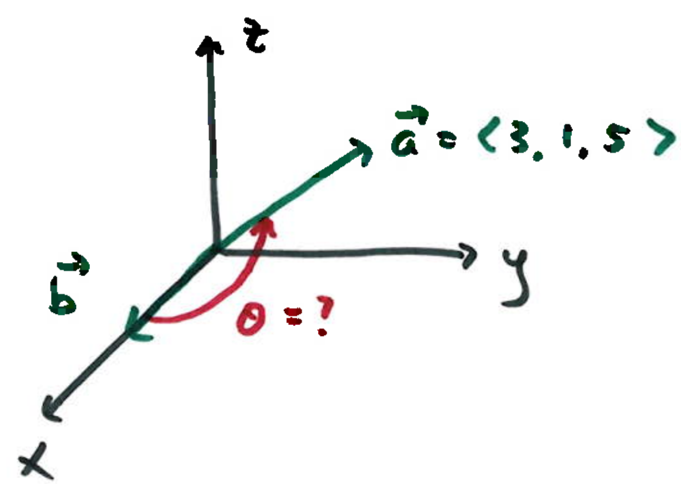
$\langle 1, 2, -2 \rangle \cdot \langle 6, 0, -8 \rangle = \sqrt{1^2+2^2+(-2)^2}\,\sqrt{6^2+0^2+(-8)^2}\,\cos\theta$

$6+0+16 = 3\cdot 10\cdot \cos\theta$

$22 = 30\cos\theta$

$\cos\theta = \dfrac{22}{30}$ $\qquad \theta = \cos^{-1}\left(\dfrac{22}{30}\right) \approx 43^\circ$

<u>example</u> What is the angle between $\vec{a} = \langle 3, 1, 5 \rangle$ and the positive x-axis ?



pick any vector along positive x-axis to be the second vector, then find angle between them

$\vec{b} = \langle 1, 0, 0 \rangle$ (but any $\langle k, 0, 0 \rangle$ works)

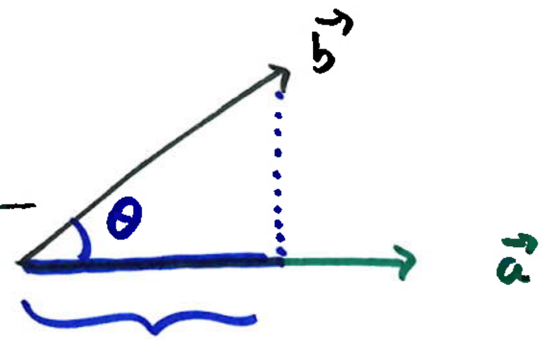
(k does not affect angle between the vectors)

$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$

$\langle 3, 1, 5 \rangle \cdot \langle 1, 0, 0 \rangle = \sqrt{35}\sqrt{1} \cos\theta$

$3 = \sqrt{35} \cos\theta$ $\qquad \cos\theta = \frac{3}{\sqrt{35}}$ direction cosine

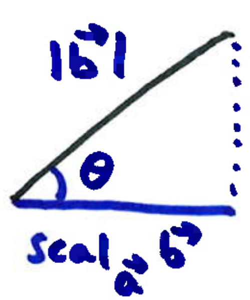
$\theta \approx 60^\circ$ direction angle the dot product can give us the <u>projection</u> of one vector onto another



"shadow" of $\vec{b}$ on $\vec{a}$

the length of this "shadow" is called the <u>scalar projection</u> of $\vec{b}$ onto $\vec{a}$

written as $\quad \text{scal}_{\vec{a}}\vec{b}$



from basic trig, $\quad \cos\theta = \dfrac{\text{scal}_{\vec{a}}\vec{b}}{|\vec{b}|}$

$$\text{scal}_{\vec{a}}\vec{b} = |\vec{b}|\cos\theta$$

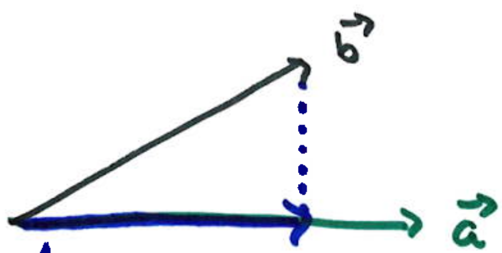
from $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ we get $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

replace the $\cos\theta$ on bottom of last page

$$\mathrm{scal}_{\vec{a}}\vec{b} = |\vec{b}|\cos\theta$$

$$= \cancel{|\vec{b}|}\,\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cancel{|\vec{b}|}}$$

$$\boxed{\mathrm{scal}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$$



↳ vector projection of $\vec{b}$ onto $\vec{a}$: vector in same direction as $\vec{a}$ w/ magnitude $\mathrm{scal}_{\vec{a}}\vec{b}$

to give it the direction of $\vec{a}$ w/o changing magnitude,

we use a unit vector pointing in same direction as $\vec{a}$

so, the vector projection of $\vec{b}$ onto $\vec{a}$ is

$$\text{proj}_{\vec{a}}\vec{b} = \underbrace{\text{scal}_{\vec{a}}\vec{b}}_{\text{magnitude}}\ \underbrace{\frac{\vec{a}}{|\vec{a}|}}_{\text{direction (as a unit vector)}}$$

$$\boxed{\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\,\frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\,\vec{a}}$$

<u>example</u> $\vec{a} = \langle 1, 2 \rangle \qquad \vec{b} = \langle 3, -4 \rangle$

$$\text{scal}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{-5}{\sqrt{5}} = -\sqrt{5}$$

negative because

$$\text{proj}_{\vec{a}}\vec{b} = \text{scal}_{\vec{a}}\vec{b}\,\frac{\vec{a}}{|\vec{a}|} = -\sqrt{5}\,\frac{\langle 1, 2\rangle}{\sqrt{5}} = -\langle 1, 2 \rangle = \langle -1, -2 \rangle$$

$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$

notice if $\vec{a} \perp \vec{b}$, then $\theta = 90^\circ = \pi/2$ and $\cos\theta = 0$

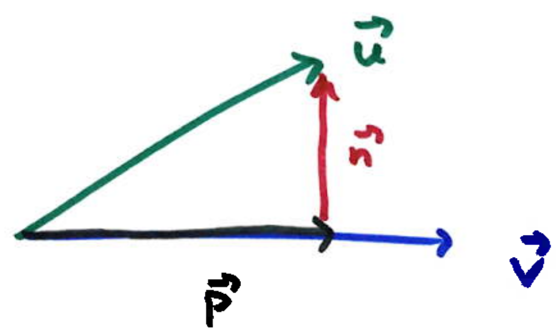
so, if $\vec{a} \perp \vec{b}$ then $\vec{a} \cdot \vec{b} = 0$

<u>example</u>

$\vec{u} = \langle -1, -4, -1 \rangle \quad \vec{v} = \langle 2, -2, 3 \rangle$

express $\vec{u}$ as $\vec{u} = \vec{p} + \vec{n}$ where $\vec{p}$ is parallel to $\vec{v}$

and $\vec{n}$ is perpendicular to $\vec{v}$



$\vec{p}$ is $\text{proj}_{\vec{v}}\vec{u}$ = (see examples)

$= \ldots$

$= \left\langle \frac{6}{17}, -\frac{6}{17}, \frac{9}{17} \right\rangle$

find $\vec{n}$: $\vec{u} = \vec{p} + \vec{n}$ so $\vec{n} = \vec{u} - \vec{p}$

$\vec{n} = \langle -1, -4, -1 \rangle - \left\langle \frac{6}{17}, -\frac{6}{17}, \frac{9}{17} \right\rangle = \left\langle -\frac{23}{17}, -\frac{62}{17}, \frac{-26}{17} \right\rangle$