11.2 Properties of Power Series (part 2)

last time : $\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ converges $|x|<1$

reuse it for similar expressions

change x to $-x^2$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

$$= \sum_{k=0}^{\infty} (-x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

converges if

$|-x^2|<1$

$\hookrightarrow |x^2|<1$

today: generate more power series by differentiating or integrating "model series" $\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

example $g(x) = \frac{1}{(1+x^2)^2}$ find power series representation

re-use $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, we have $\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}$

notice $\frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{-2x}{(1+x^2)^2} = -2x \cdot \frac{1}{(1+x^2)^2}$ ← $g(x)$ power series?

we get : $\frac{1}{(1+x^2)^2} = -\frac{1}{2x} \frac{d}{dx}\left(\frac{1}{1+x^2}\right)$ ← we have power series

$$\frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} (-1)^k x^{2k}\right) = \frac{d}{dx}\left(1 - x^2 + x^4 - x^6 + x^8 - \dots\right)$$

$$= -2x + 4x^3 - 6x^5 + 8x^7 - 10x^9 + 12x^{11} - \dots$$

$$-\frac{1}{2x}\frac{d}{dx}\left(\frac{1}{1+x^2}\right) = -\frac{1}{2x}\left(-2x + 4x^3 - 6x^5 + 8x^7 - 10x^9 + 12x^{11} - \dots\right)$$

$= 1-2x^2+3x^4-4x^6+5x^8-6x^{10}+7x^{12}-\dots$ this is the power series of $\frac{1}{(1+x^2)^2}$

let's put into summation notation

patterns: alternating

coefficients go up by 1

powers are even

$= 1-2x^2+3x^4-4x^6+5x^8-6x^{10}+7x^{12}-\dots$ choose to start at $k=1$

$k=1 \quad k=2 \quad k=3 \quad k=4 \quad k=5 \quad k=6$

$$= \boxed{\sum_{k=1}^{\infty} (-1)^{k-1} \cdot k \cdot x^{2k-2}}$$

$2k-2$ (mistake in lecture)

differentiation / integration does NOT change the radius of convergence of the "model series"

So, this series is based on

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad |x|<1$$

so the power series we got still requires $|x|<1$, but the end behaviors can change ($x=1$, $x=-1$)

example $\ln\sqrt{16-x^2}$

model series : $\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad |x|<1$

$$\ln(16-x^2)^{1/2} = \frac{1}{2}\ln(16-x^2)$$

what happens if we differentiate $\frac{1}{2}\ln(16-x^2)$?

$$\boxed{\frac{d}{dx}\left[\frac{1}{2}\ln(16-x^2)\right]} = \frac{1}{2}\cdot\frac{-2x}{16-x^2} = \boxed{-x\cdot\frac{1}{16-x^2}}$$

use $\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

$$\frac{1}{16-x^2} = \frac{1}{16\left(1-\frac{x^2}{16}\right)} = \frac{1}{16}\cdot\frac{1}{1-\left(\frac{x}{4}\right)^2}$$

change x to $\left(\frac{x}{4}\right)^2$

$$= \frac{1}{16}\sum_{k=0}^{\infty}\left[\left(\frac{x}{4}\right)^2\right]^k = \frac{1}{16}\sum_{k=0}^{\infty}\frac{x^{2k}}{4^{2k}}$$

$$-x\cdot\frac{1}{16-x^2} = -x\cdot\frac{1}{16}\sum_{k=0}^{\infty}\frac{x^{2k}}{4^{2k}} = \frac{d}{dx}\left[\overbrace{\frac{1}{2}\ln(16-x^2)}^{\text{what we want}}\right]$$

from two lines above

$$\text{so } \frac{1}{2}\ln(16-x^2) = \ln\sqrt{16-x^2} = \int \frac{-x}{16}\sum_{k=0}^{\infty}\frac{x^{2k}}{4^{2k}}\,dx$$

$$= \int -\underset{4^2}{\frac{x}{16}}\left(1 + \frac{x^2}{4^2} + \frac{x^4}{4^4} + \frac{x^6}{4^6} + \frac{x^8}{4^8} + \cdots\right) dx$$

$$= \int \left(-\frac{x}{4^2} - \frac{x^3}{4^4} - \frac{x^5}{4^6} - \frac{x^7}{4^8} - \frac{x^9}{4^{10}} - \cdots\right) dx$$

$$= -\frac{x^2}{2\cdot 4^2} - \frac{x^4}{4\cdot 4^4} - \frac{x^6}{6\cdot 4^6} - \frac{x^8}{8\cdot 4^8} - \frac{x^{10}}{10\cdot 4^{10}} - \frac{x^{12}}{12\cdot 4^{12}} - \cdots + C$$

$$= \ln\sqrt{16-x^2}$$

what is C?

at $x=0$, $\ln\sqrt{16-x^2} = \underbrace{-\frac{x^2}{2\cdot 4^2} - \frac{x^4}{4\cdot 4^4} - \cdots}_{0} + C$

$$\ln\sqrt{16} = 0 + C$$

$$\ln 4 = C$$

so, $$\boxed{\ln\sqrt{16-x^2} = \underset{C}{\ln 4} - \left(\frac{x^2}{2\cdot 4^2} + \frac{x^4}{4\cdot 4^4} + \frac{x^6}{6\cdot 4^6} + \frac{x^8}{8\cdot 4^8} + \cdots\right)}$$

$$\ln 4 - \left(\underset{k=1}{\frac{x^2}{2\cdot 4^2}} + \underset{k=2}{\frac{x^4}{4\cdot 4^4}} + \underset{k=3}{\frac{x^6}{6\cdot 4^6}} + \underset{k=4}{\frac{x^8}{8\cdot 4^8}} + \cdots \right)$$

choose to start at $k = 1$

$$= \ln 4 - \sum_{k=1}^{\infty} \frac{x^{2k}}{2k\cdot 4^{2k}}$$

example what function is represented by $\sum_{k=0}^{\infty} \frac{(x-2)^k}{3^{2k}}$?

again, use $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

$$\sum_{k=0}^{\infty} \left(\frac{x-2}{3^2}\right)^k$$

change x to $\frac{x-2}{3^2}$

so, $\sum_{k=0}^{\infty} \left(\frac{x-2}{3^2}\right)^k = \frac{1}{1-\left(\frac{x-2}{9}\right)} = \frac{9}{9-(x-2)} = \boxed{\frac{9}{11-x}}$