11.3 Taylor Series

Taylor series: $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$

a: center (where the Taylor series is built)

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

when $a=0$ we call the resulting Taylor Series the Maclaurin Series

Maclaurin series of $f(x)$: $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$

$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

example Maclaurin series of $f(x) = \sin x$

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$

pattern repeats after 4 derivatives

all even derivs are 0

all odd derivs are ± 1

Maclaurin series of $\sin x$

$$f(x) = f^{0}(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$= \underbrace{x}_{k=0} - \underbrace{\frac{1}{3!}x^3}_{k=1} + \underbrace{\frac{1}{5!}x^5}_{k=2} - \underbrace{\frac{1}{7!}x^7}_{k=3} + \dots$$

pack into summation

$$= \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}}$$

near $x=0$, $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

interval of convergence of $\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

Ratio Test: $\lim_{k\to\infty} \left| \frac{(-1)^{k+1} \frac{x^{2(k+1)+1}}{(2(k+1)+1)!}}{(-1)^k \frac{x^{2k+1}}{(2k+1)!}} \right|$

$$= \lim_{k\to\infty} \left| \frac{x^{2k+3}}{(2k+3)!} \cdot \frac{(2k+1)!}{x^{2k+1}} \right|$$

$$= \lim_{k\to\infty} \left| x^2 \cdot \frac{\cancel{(2k+1)}\cancel{(2k)}\cancel{(2k-1)}\cancel{(2k-2)}\cancel{\cdots}\cancel{(1)}}{(2k+3)(2k+2)\cancel{(2k+1)}\cancel{(2k)}\cancel{(2k-1)}\cancel{(2k-2)}\cancel{\cdots}\cancel{(1)}} \right| = 0$$

so converge on $-\infty < x < \infty$ or $(-\infty, \infty)$

radius of convergence ∞

we can re-use $\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ for anything that resembles $\sin(x)$

for example, $\sin(2x) = \sum_{k=0}^{\infty} (-1)^k \frac{(2x)^{2k+1}}{(2k+1)!}$

$f(x) = x^3 \sin\left(\frac{x^2}{3}\right)$ converges on $(-\infty, \infty)$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\sin\left(\frac{x^2}{3}\right) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} \left(\frac{x^2}{3}\right)^{2k+1}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} \frac{x^{4k+2}}{3^{2k+1}}$$ converges on $(-\infty, \infty)$

bump up by 3

$$x^3 \sin\left(\frac{x^2}{3}\right) = x^3 \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} \frac{x^{4k+2}}{3^{2k+1}} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} \frac{x^{4k+5}}{3^{2k+1}}$$

$$= \frac{x^5}{3} - \frac{x^9}{3!\,3^2} + \frac{x^{13}}{5!\,3^5} - \cdots$$

Common Maclaurin Series   a = 0 only

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k, \quad \text{for } |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^k x^k + \cdots = \sum_{k=0}^{\infty} (-1)^k x^k, \quad \text{for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for } |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{k+1} x^k}{k} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \quad \text{for } -1 < x \le 1$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k}, \quad \text{for } -1 \le x < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^k x^{2k+1}}{2k+1} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \quad \text{for } |x| \le 1$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

example $f(x) = \dfrac{e^x - 1}{5x}$ Maclaurin series ?

from Table: $e^x = \displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!} \quad (-\infty, \infty)$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^x - 1 = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \right) - 1$$

$$= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{k=1}^{\infty} \frac{x^k}{k!}$$

$$\frac{e^x - 1}{5x} = \frac{1}{5x}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \right) = \frac{1}{5x} \sum_{k=1}^{\infty} \frac{x^k}{k!}$$

$$= \frac{1}{5} + \frac{x}{5\cdot 2!} + \frac{x^2}{5\cdot 3!} + \frac{x^4}{5\cdot 4!} + \cdots = \sum_{k=1}^{\infty} \frac{x^{k-1}}{5\cdot k!}$$

The Table we saw is a list of <u>Maclaurin</u> series $\rightarrow$ $a=0$ only

ANY other a, find Taylor series by differentiating

<u>example</u> $f(x) = \sin x \quad a = \frac{\pi}{6}$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$f(x) = \sin x \qquad f\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$f'(x) = \cos x \qquad f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$f''(x) = -\sin x \qquad f''\left(\frac{\pi}{6}\right) = -\frac{1}{2}$

$f'''(x) = -\cos x \qquad f'''\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

$f^{(4)}(x) = \sin x \qquad f^{(4)}\left(\frac{\pi}{6}\right) = \frac{1}{2}$

cycle repeats

near $x = \frac{\pi}{6}$, $\sin x$ can be approximated as

$$\sin x = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right) - \frac{1/2}{2!}\left(x-\frac{\pi}{6}\right)^2 - \frac{\sqrt{3}/2}{3!}\left(x-\frac{\pi}{6}\right)^3 + \frac{1/2}{4!}\left(x-\frac{\pi}{6}\right)^4 + \cdots$$

in practice, we chop it off after some k, then look at the error

Remainder of k^{th}-order Taylor polynomial is

$$R_k = \frac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1}$$

for $\sin x$ with $a = \frac{\pi}{6}$

$$R_k = \frac{\pm\cos(c)}{(k+1)!}\left(x-\frac{\pi}{6}\right)^{k+1} \quad \text{or} \quad \frac{\pm\sin(c)}{(k+1)!}\left(x-\frac{\pi}{6}\right)^{k+1}$$

how do we know as $k\to\infty$, the Taylor series = true function?

$\rightarrow$ if $\lim_{k\to\infty} R_k = 0$

between -1 and 1

$$R_k = \frac{\pm\cos(c)}{(k+1)!}\left(x-\frac{\pi}{6}\right)^{k+1}$$

it's clear that $R_k \to 0$ as $k\to\infty$
(same for $\pm\sin(c)$ on top)

to ∞ really quickly