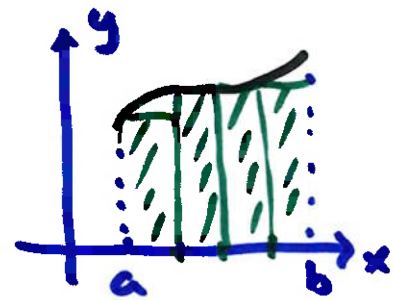
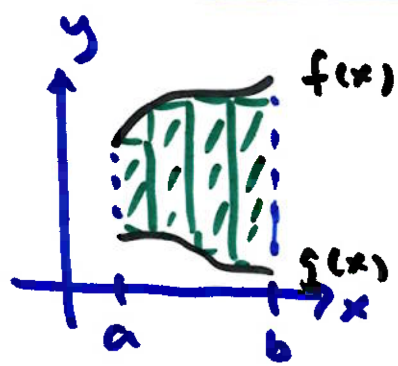
12.3 Areas and Lengths in Polar Coordinates

In Cartesian, $\int_a^b f(x)\,dx$



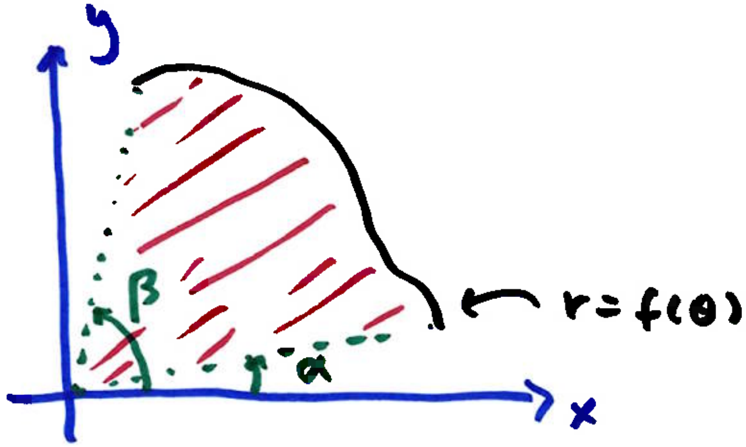
shrink rectangles then sum infinitely-many of them



$$\int_a^b [f(x)-g(x)]\,dx$$

Sum of infinitely-many rectangles between f(x) and g(x)

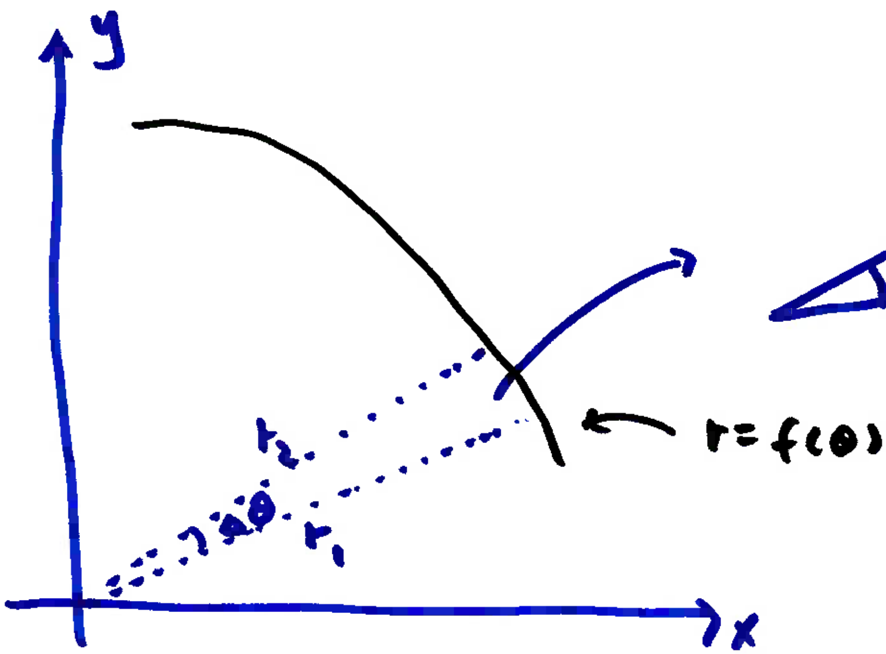
In Polar, same idea, but instead of rectangles, we use thin slices of circles

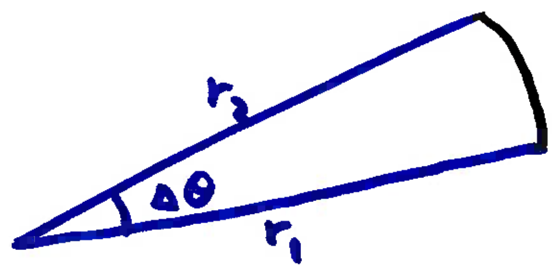


$r = f(\theta)$ curve in polar equation

area of red region ?

divide into thin slices





when $\Delta\theta$ is small, $r_1 \approx r_1 = r$

from geometry, the area of this segment is

$\boxed{\frac{1}{2} r^2 \Delta\theta}$ this is the polar equivalent of a rectangle

Sum up these slices from α to β, and shrink $\Delta\theta \to d\theta$

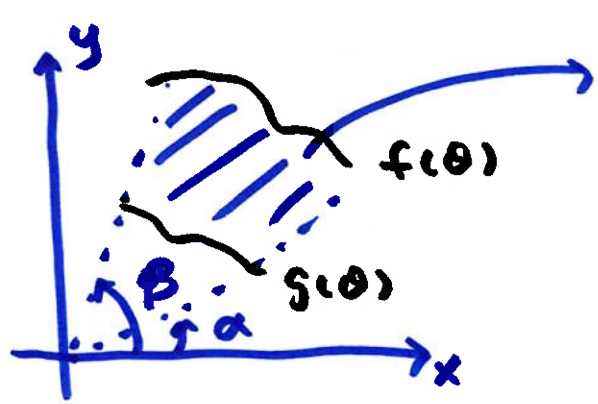
so, the entire region has area $\boxed{\int_\alpha^\beta \frac{1}{2} r^2 \, d\theta}$

function of θ

if $r = f(\theta)$

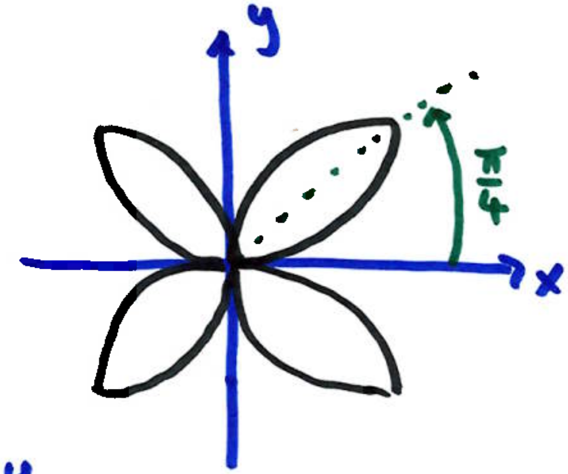
$$\boxed{\int_\alpha^\beta \frac{1}{2} [f(\theta)]^2 \, d\theta}$$

between curves

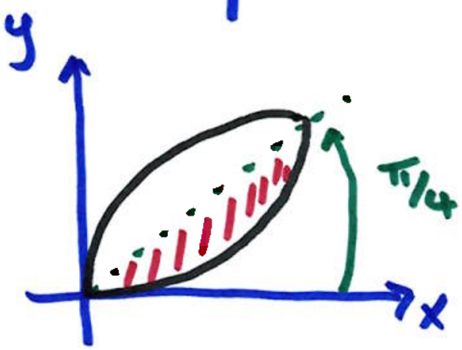


$$\int_\alpha^\beta \left\{ \frac{1}{2} [f(\theta)]^2 - \frac{1}{2} [g(\theta)]^2 \right\} d\theta$$

example Find area of one petal of the rose $r = \sin 2\theta$



find area of any. for simplicity, let's find QI



notice symmetry, so we can find area of half (red portion) then double it

$$\int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

$\sin 2\theta$

double

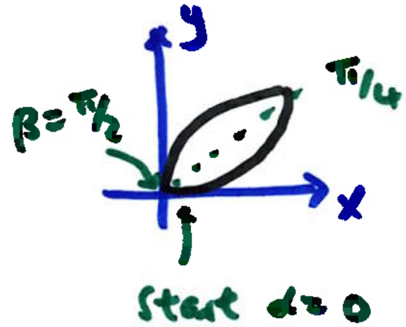
to middle of petal

$$2 \int_0^{\pi/4} \frac{1}{2} (\sin 2\theta)^2 \, d\theta = \int_0^{\pi/4} \sin^2 2\theta \, d\theta = \int_0^{\pi/4} \frac{1 - \cos(4\theta)}{2} \, d\theta$$

minus

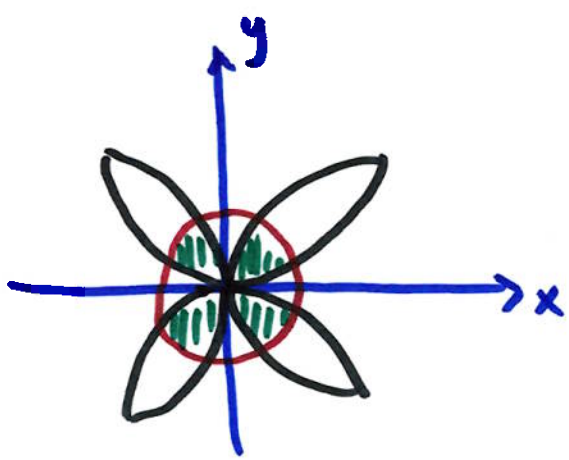
$$= \frac{1}{2} \int_0^{\pi/4} 1 - \cos(4\theta) \, d\theta = \frac{1}{2} \left(\theta + \frac{1}{4} \sin 4\theta \right) \Big|_0^{\pi/4} = \frac{1}{2} \left(\frac{\pi}{4} - 0 \right) = \boxed{\frac{\pi}{8}}$$

alternative : ~~no~~ w/o using symmetry

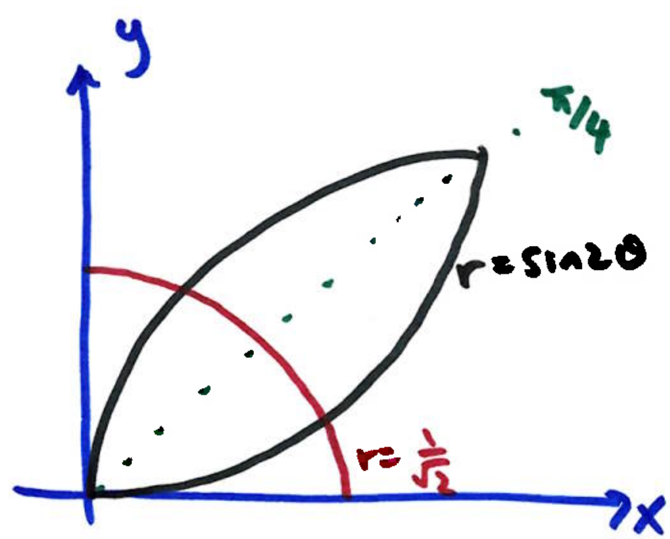


$$\int_0^{\pi/2} \frac{1}{2}(\sin 2\theta)^2\, d\theta = \cdots = \frac{\pi}{8}$$

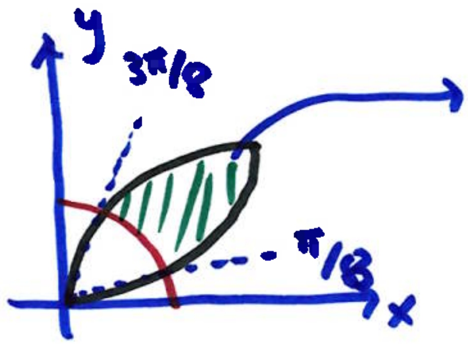
<u>example</u> Area ~~inside~~ bounded by $\underbrace{r=\frac{1}{\sqrt{2}}}_{\text{circle radius } \frac{1}{\sqrt{2}}}$ and ~~out~~ $\underbrace{r=\sin 2\theta}_{\text{rose we looked at}}$ closer to origin



QI :



one approach: find area of rose petal outside circle, then subtract from area of one petal



$$\int_{\pi/8}^{3\pi/8} \left[\underbrace{\tfrac{1}{2}(\sin 2\theta)^2 \cancel{d\theta}}_{\substack{\text{outside}\\ \text{(rose)}}} - \underbrace{\tfrac{1}{2}\left(\tfrac{1}{\sqrt{2}}\right)^2}_{\substack{\text{inside}\\ \text{(circle)}}} \right] d\theta$$

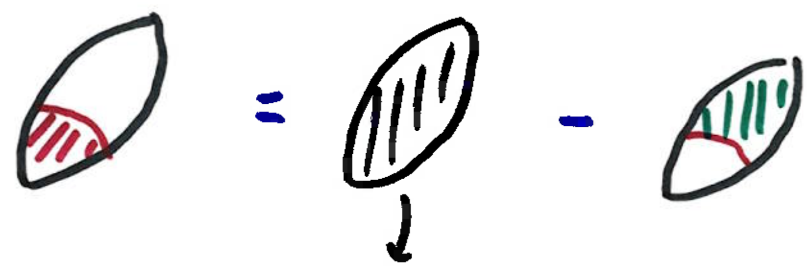
intersection: $r = \frac{1}{\sqrt{2}}$ and $r = \sin 2\theta$ are equal

Solve: $\frac{1}{\sqrt{2}} = \sin 2\theta$

$\vdots$

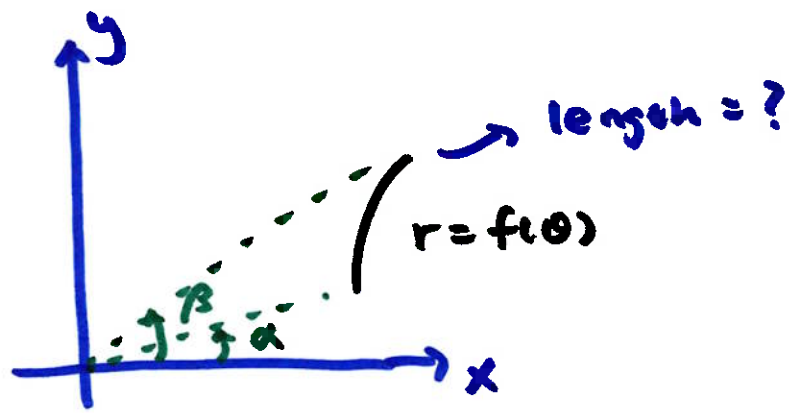
$\theta = \frac{\pi}{8}$

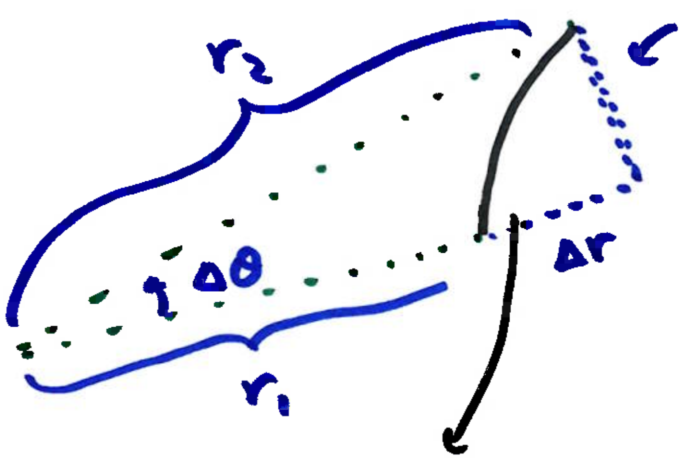
then, subtract from rose petal area



last example

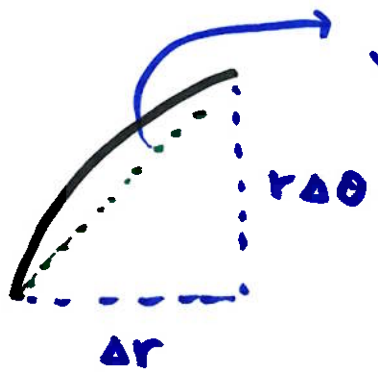
Arc length :





circular arc length $= r\Delta\theta$

when $\Delta\theta$ is small, $r_1 \approx r_2 = r$



$\sqrt{(\Delta r)^2 + (r\Delta\theta)^2} \approx$ length of black curve

$\hookrightarrow \sqrt{(\Delta\theta)^2\left[\frac{(\Delta r)^2}{(\Delta\theta)^2} + r^2\right]}$

$$= \sqrt{\left[\left(\frac{\Delta r}{\Delta \theta}\right)^2 + r^2\right](\Delta\theta)^2} = \sqrt{r^2 + \left(\frac{\Delta r}{\Delta \theta}\right)^2}\ \Delta\theta$$

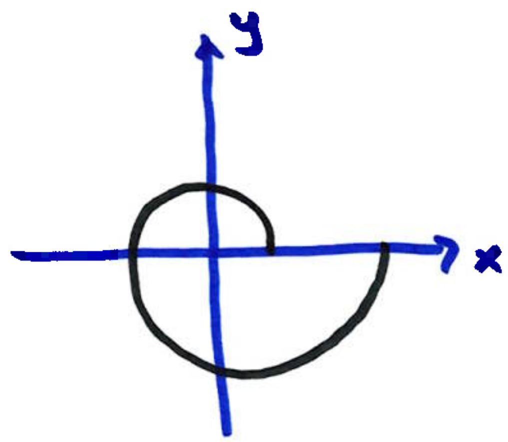
shrink : $\dfrac{\Delta r}{\Delta \theta} \to \dfrac{dr}{d\theta}$

$\Delta\theta \to d\theta$

accumulate from $\theta = \alpha$ to $\theta = \beta$ by integration

$$\boxed{\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta}$$

<u>example</u> Length of $r = e^{\theta}$ $\quad 0 \le \theta \le 2\pi$



logarithmic spiral

length: $\displaystyle\int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$r = e^{\theta}$

$\dfrac{dr}{d\theta} = e^{\theta}$

$$\int_0^{2\pi} \sqrt{(e^{\theta})^2 + (e^{\theta})^2}\, d\theta = \int_0^{2\pi} \sqrt{e^{2\theta} + e^{2\theta}}\, d\theta = \int_0^{2\pi} \sqrt{2e^{2\theta}}\, d\theta$$

$(e^{2\theta})^{1/2}$

$$= \int_0^{2\pi} \sqrt{2} \cdot \sqrt{e^{2\theta}}\, d\theta = \int_0^{2\pi} \sqrt{2}\, e^{\theta}\, d\theta = \sqrt{2} \int_0^{2\pi} e^{\theta}\, d\theta$$

$$= \sqrt{2}\, (e^{\theta}) \Big|_0^{2\pi} = \boxed{\sqrt{2}\, (e^{2\pi} - 1)}$$