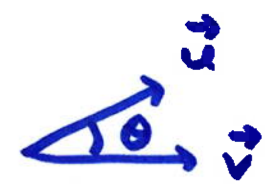
13.4 The Cross Product

another way vectors multiply



dot product : $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$

$\vec{u} = \langle a, b, c \rangle \quad \vec{v} = \langle d, e, f \rangle$

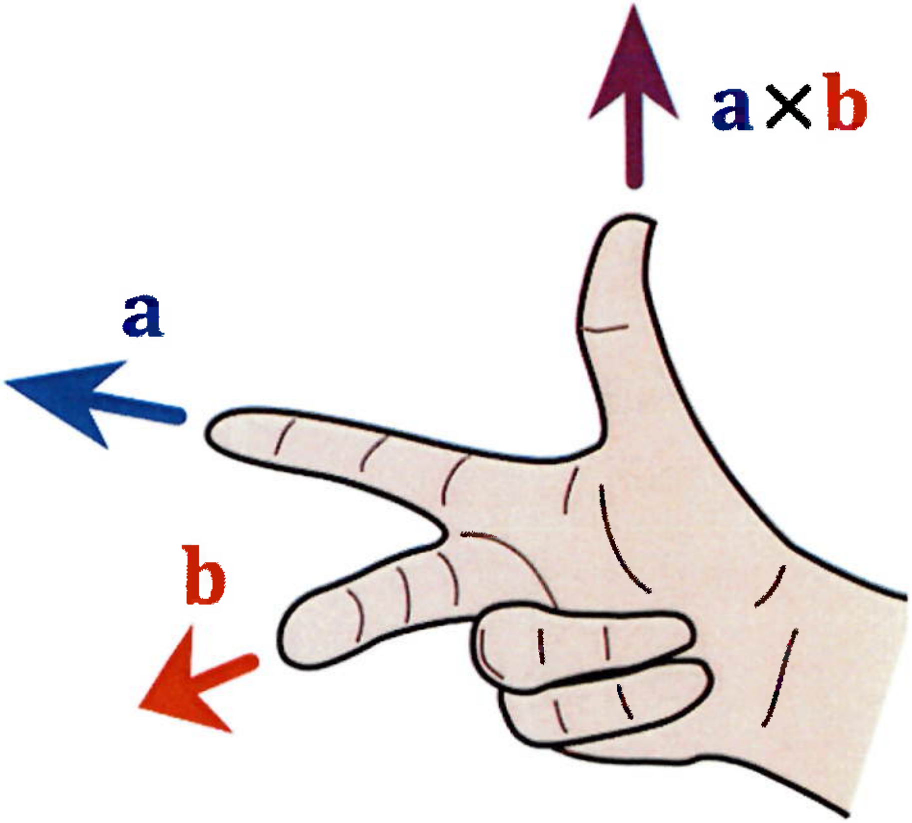
$\vec{u} \cdot \vec{v} = ad + be + cf$ <u>scalar</u>

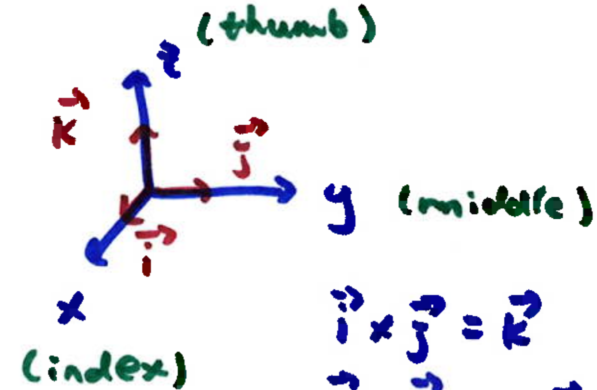
the cross product of $\vec{u}, \vec{v}$ is $\vec{u} \times \vec{v}$ (or $\vec{v} \times \vec{u}$)

the result is a <u>vector</u>

magnitude : $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$

direction : by the right-hand rule

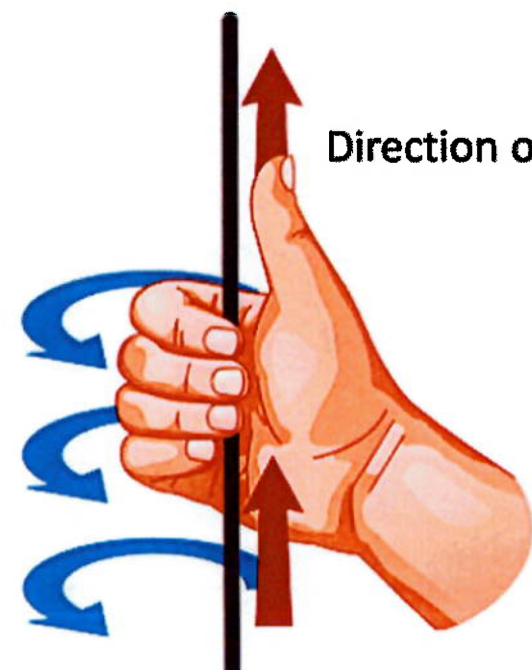


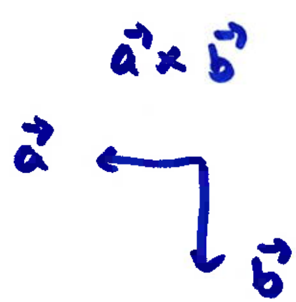


$\vec{i} \times \vec{j} = \vec{k}$

$\vec{j} \times \vec{i} = -\vec{k}$

CURL RIGHT HAND RULE





$\vec{a} \times \vec{b}$: out of screen / paper

$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$ in cross product, order matters

how to compute $\vec{u} \times \vec{v}$?

one way: use $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$ to find magnitude

then right-hand rule for direction

the other (algebraic) way: the <u>determinant</u> of a special matrix

2x2 matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ determinant is $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

$= (1)(4) - (2)(3)$

$= 4 - 6 = -2$

$$\begin{vmatrix} -3 & 1 \\ 7 & -4 \end{vmatrix} = (-3)(-4) - (1)(7) = 12 - 7 = 5$$

we need determinant of a special 3x3 matrix for cross product

$\vec{u} = \langle 2, 1, 2 \rangle \quad \vec{v} = \langle 5, 0, 1 \rangle$

find $\vec{u} \times \vec{v}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 2 \\ 5 & 0 & 1 \end{vmatrix}$$

first row: $\vec{i}, \vec{j}, \vec{k}$

2nd row: first vector in cross product

3rd row: the second vector

move along first row

$\vec{j}$ gets negtive

$$= \vec{i} \underbrace{\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}}_{\text{determinant of leftover after removing row and column of } \vec{i}} - \vec{j} \underbrace{\begin{vmatrix} 2 & 2 \\ 5 & 1 \end{vmatrix}}_{\text{determinant of leftover after removing row, column } \vec{j} \text{ is in}} + \vec{k} \begin{vmatrix} 2 & 1 \\ 5 & 0 \end{vmatrix}$$

$$= \vec{i}(1) - \vec{j}(-8) + \vec{k}(-5) = \vec{i} + 8\vec{j} - 5\vec{k} = \langle 1, 8, -5 \rangle$$

$\vec{u} = \langle 2, 1, 2 \rangle \quad \vec{v} = \langle 5, 0, 1 \rangle$

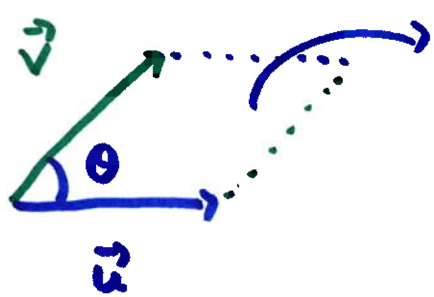
$\vec{u} \times \vec{v} = \langle 1, 8, -5 \rangle \qquad \vec{v} \times \vec{u} = -\langle 1, 8, -5 \rangle = \langle -1, -8, 5 \rangle$ (check on your own)

notice $(\vec{u} \times \vec{v}) \cdot \vec{u} = \langle 1, 8, -5 \rangle \cdot \langle 2, 1, 2 \rangle = 2 + 8 - 10 = 0$

$(\vec{u} \times \vec{v}) \cdot \vec{v} = \langle 1, 8, -5 \rangle \cdot \langle 5, 0, 1 \rangle = 5 + 0 - 5 = 0$

this means, $\vec{u} \times v$ (or $\vec{v} \times \vec{u}$) is perpendicular or orthogonal to BOTH $\vec{u}$ and $\vec{v}$

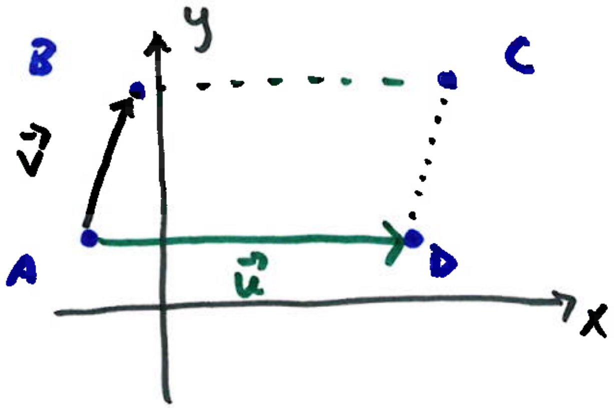
if $\vec{u}$ and $\vec{v}$ form the two sides of a parallelogram sharing a corner, then we can also show that $|\vec{u} \times \vec{v}|$ = area of the parallelogram



$$\text{area} = |\vec{u} \times \vec{v}|$$
$$= |\vec{u}||\vec{v}| \sin\theta$$

example Find area of ~~parra~~ parallelogram with vertices

$A(-3, 4)$, $B(-1, 7)$, $C(3, 5)$, $D(1, 2)$



$\vec{u} = \langle 4, -2 \rangle = \langle 4, -2, 0 \rangle$

$\vec{v} = \langle 2, 3 \rangle = \langle 2, 3, 0 \rangle$

area of p-gram is $|\vec{u} \times \vec{v}|$ or $|\vec{v} \times \vec{u}|$

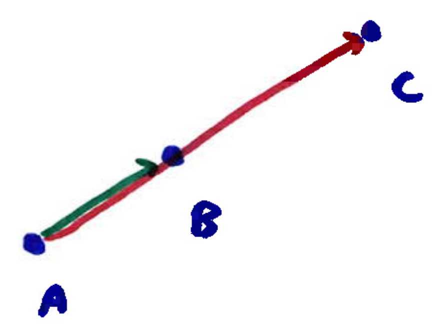
$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -2 & 0 \\ 2 & 3 & 0 \end{vmatrix} = \vec{i} \begin{vmatrix} -2 & 0 \\ 3 & 0 \end{vmatrix} - \vec{j} \begin{vmatrix} 4 & 0 \\ 2 & 0 \end{vmatrix} + \vec{k} \begin{vmatrix} 4 & -2 \\ 2 & 3 \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(16) = 16\vec{k} = \langle 0, 0, 16 \rangle$$

$|\vec{u} \times \vec{v}| = \boxed{16}$ area of the p-gram

or $|\vec{u}||\vec{v}|\sin\theta$

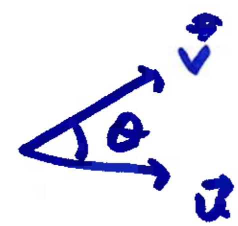
collinear vectors: along the same line



if $\vec{AB} \parallel \vec{AC}$, then angle between

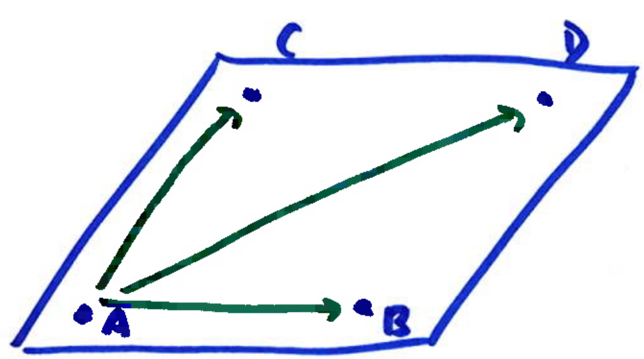
is $\theta = 0$ (same direction)

$= 180^\circ$ (opposite direction)



and $\sin\theta = 0$

therefore, if $\vec{AB} \parallel \vec{AC}$, then $|\vec{AB} \times \vec{AC}| = 0$



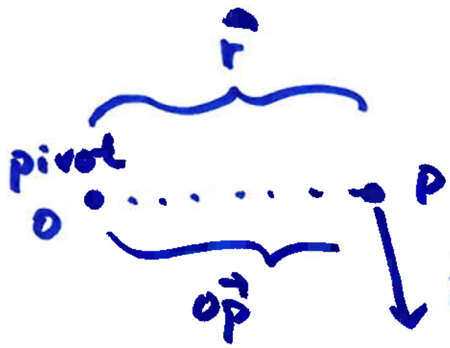
A, B, C, D are points on a plane

$\vec{AB} \times \vec{AD}$, $\vec{AD} \times \vec{AB}$, $\vec{AB} \times \vec{AC}$, $\vec{AC} \times \vec{AB}$

$\vec{AD} \times \vec{AC}$, $\vec{AC} \times \vec{AD}$

are ALL orthogonal to the plane

cross product is orthogonal to BOTH parent vectors physical applications: torque or moment can be found by cross product



$\vec{F}$: force applied at P at distance r from pivot

the torque about pivot is $\vec{OP} \times \vec{F}$

magnetism: $\vec{F} = q\,(\vec{v} \times \vec{B})$

q: charge

$\vec{v}$: flow of current

$\vec{B}$: magnetic field