15.3 Double Integrals over General Regions

last time: \[ \iint_R f(x,y) \, dA \quad R = [a,b] \times [c,d] \]
\[ \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy \]

today: \[ \iiint_D f(x,y) \, dA \quad \text{where } D \text{ is NOT a rectangle.} \]

**Example**: Volume of solid under \( f(x, y) = xy^2 \)
over the region \( D = \{(x,y) \mid 0 \leq x \leq \sqrt{2}, x^2 \leq y \leq 2\} \)

\[ x = 2 \quad \text{variable w/ constant bounds has to be done LAST} \]
\[ V = \int_0^{\sqrt{2}} \int_{x^2}^2 xy^2 \, dy \, dx \]

\[ = \int_0^{\sqrt{2}} x \cdot \frac{y^3}{3} \bigg|_{y=x^2}^{y=2} \, dx \]

\[ = \int_0^{\sqrt{2}} \left( \frac{8x^5}{3} - \frac{x^7}{3} \right) \, dx = \frac{4x^6}{3} - \frac{x^8}{24} \bigg|_0^{\sqrt{2}} = \boxed{2} \]

\[ \int_0^{\sqrt{2}} \int_{x^2}^2 xy^2 \, dy \, dx \neq \int_{x^2}^2 \int_0^{\sqrt{2}} xy^2 \, dx \, dy \]

but we could still switch order by re-defining \( D \) describing
D = \{(x,y) \mid 0 \leq x \leq \sqrt{2}, \quad x^2 \leq y \leq 2\} \quad \text{Type I region} \\
\rightarrow x \text{ bounded by constants}

\begin{align*}
&y = 2 \\
&y = x^2 \\
&x = \sqrt{y} \\
&x = 0
\end{align*}

D = \{(x,y) \mid 0 \leq y \leq 2, \quad 0 \leq x \leq \sqrt{y}\} \quad \text{Type II region}

\begin{align*}
&\text{bottom} \\
&\text{top} \\
&\text{left} \\
&\text{right}
\end{align*}
\[ V = \int_0^2 \int_0^{\sqrt{3}} xy^2 \, dx \, dy \]
\[ = \int_0^2 \frac{x^2}{2} y^2 \left| \right. \bigg|_{x=0}^{x=\sqrt{3}} \, dy = \int_0^2 \frac{1}{2} y^3 \, dy \]
\[ = \frac{1}{8} y^4 \bigg|_0^2 = \boxed{2} \]
Example \( \iint_{D} y \, dA \)

\( D \) is bounded by

\( y = x - 2 \) and \( x = y^2 \)

Type II is easier to formulate
(type I is harder because top and bottom are the same curve at the end left end of region)

Constant bounds for \( y \): \( x = y + 2 \) equal to \( x = y^2 \)

\( y^2 = y + 2 \)
\( y^2 - y - 2 = 0 \)
\( (y - 2)(y + 1) = 0 \)
\( y = -1, \quad y = 2 \)
\[ D = \{ (x,y) \mid -1 \leq y \leq 2, \quad y^2 \leq x \leq y+2 \} \]

\[
V = \int_{-1}^{2} \int_{y^2}^{y+2} y \, dx \, dy \\
= \int_{-1}^{2} y \left[ x \right]_{x=y^2}^{x=y+2} \, dy \\
= \int_{-1}^{2} y (y+2-y^2) \, dy \\
= \int_{-1}^{2} (y^2+2y-y^3) \, dy \\
= \frac{9}{4}
\]
Example

Volume of solid under \( f(x,y) = xy \) above the triangle with vertices \((0,0), (3,0), (0,1)\)

**Type I**

\[ D = \{(x,y) \mid 0 \leq x \leq 3, \ 0 \leq y \leq 1 - \frac{1}{3}x\} \]

\[ y = 1 - \frac{1}{3}x \]

\[ y = 0 \]

\[ x = 3 - 3y \]

\[ \frac{1}{3}x = 1 - y \]

**Type II**

\[ D = \{(x,y) \mid 0 \leq y \leq 1, \ 0 \leq x \leq 3 - 3y\} \]

\[ x = 0 \]

\[ x = 3 - 3y \]
\[ V = \int_0^1 \int_0^{3-3y} xy \, dx \, dy \]
\[ = \int_0^1 \left. \frac{1}{2} x^2 y \right|_{x=0}^{x=3-3y} \, dy \]
\[ = \int_0^1 \frac{1}{2} (3-3y)^2 y \, dy = \int_0^1 \frac{1}{2} (9 - 18y + 9y^2) y \, dy \]
\[ = \frac{1}{2} \int_0^1 9y - 18y^2 + 9y^3 \, dy = \frac{1}{2} \left( \frac{9}{2} y^2 - 6y^3 + \frac{9}{4} y^4 \right) \bigg|_0^1 \]
\[ = \frac{3}{8} \]
Average value of \( f(x,y) \) over \( D \)

Recall the average value of \( y = f(x) \) on \( a \leq x \leq b \) is

\[
    f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) \, dx
\]

If \( z = f(x,y) \) over \( D \), the average of \( f(x,y) \) over \( D \) is

\[
    f_{\text{avg}} = \frac{1}{\text{area of } D} \iint_D f(x,y) \, dA
\]
Example: average of \( f(x,y) = xy \) over the triangle in the last example

\[
\text{area of } D = \frac{1}{2} \cdot 3 \cdot 1 = \frac{3}{2}
\]

\[
\iint_D f(x,y) \, dA = \frac{3}{8}
\]

\[
f_{\text{avg}} = \frac{1}{3/2} \left( \frac{3}{8} \right) = \frac{1}{4}
\]