15.9 Triple Integrals in Spherical Coordinates

location given by \((\rho, \theta, \phi)\)

\(\rho\): distance to point from origin \((\rho \geq 0)\)

\(\theta\): angle from the positive x-axis to the projection of point onto xy-plane (same as \(\theta\) in polar or cylindrical) \(0 \leq \theta \leq 2\pi\)

\(\phi\): angle measured from positive \(\xi\)-axis to point \(0 \leq \phi \leq \pi\)
Conversion to Cartesian

(green triangle from last page)

\[ \cos \phi = \frac{z}{\rho} \rightarrow z = \rho \cos \phi \]

\[ \sin \phi = \frac{r}{\rho} \rightarrow r = \rho \sin \phi \]

on xy-plane

\[ \cos \theta = \frac{x}{r} = \frac{x}{\rho \sin \phi} \]

\[ x = \rho \sin \phi \cos \theta \]

\[ \sin \theta = \frac{y}{r} = \frac{y}{\rho \sin \phi} \]

\[ y = \rho \sin \phi \sin \theta \]

\[ x^2 + y^2 + z^2 = \rho^2 \]
Example: In spherical: \((1, \pi/3, \pi/6)\)

in Cartesian?

\[
\begin{align*}
X &= \rho \sin \phi \cos \theta = 1 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \\
Y &= \rho \sin \phi \sin \theta = 1 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4} \\
Z &= \rho \cos \phi = 1 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}
\end{align*}
\]

Check: \(X^2 + Y^2 + Z^2 = \rho^2\)

\[
\frac{1}{16} + \frac{3}{16} + \frac{3}{4} = \frac{16}{16} = 1 = \rho^2
\]

Cartesian: \(\left( \frac{1}{4}, \frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2} \right)\)
Example: In Cartesian: \((1, -1, \sqrt{2})\)
in spherical?

\[
\begin{align*}
x &= \rho \sin \phi \cos \theta \\
y &= \rho \sin \phi \sin \theta \\
z &= \rho \cos \phi \\
x^2 + y^2 + z^2 &= \rho^2
\end{align*}
\]

\[
1^2 + (-1)^2 + (\sqrt{2})^2 = \rho^2 = 4 \quad \Rightarrow \quad \rho = 2 \quad \rho \geq 0
\]

\[
\frac{y}{x} = \frac{\rho \sin \phi \sin \theta}{\rho \sin \phi \cos \theta} = \tan \theta \quad \Rightarrow \quad \frac{-1}{1} = \tan \theta 
\]

\[
\theta = \frac{7 \pi}{4}
\quad 0 \leq \theta \leq 2\pi
\]

\[
z = \rho \cos \phi \quad \Rightarrow \quad \cos \phi = \frac{z}{\rho} = \frac{\sqrt{2}}{2}
\]

\[
\phi = \frac{\pi}{4} 
\quad 0 \leq \phi \leq \pi
\]

\text{find quadrant of } \theta \text{ first!}

\theta \text{ in QIV here}
\[
\int_\mathbf{E} SSS \, dV
\]

What is \(dV\) in spherical coordinates?

If \(d\theta, d\phi, dp\) are small,

\[
psin\phi \, d\theta \quad \approx \quad \begin{array}{c}
\text{box}
\end{array}
\]

\[
p \, dp \phi \phi
\]

So

\[
dV = p^2 \sin \phi \, dp \, d\phi \, d\theta \, d\phi
\]
spherical is good when \( E \) is a sphere, part of sphere, or any sphere-like thing (cone).

Example:

\[
\int_0^6 \int_0^{\sqrt{36-x^2}} \int_0^{\frac{\sqrt{72-x^2-y^2}}{\sqrt{x^2+y^2}}} dz \, dy \, dx
\]

Bounds suggest a circle/sphere involved:

- If \( E \) is sphere-like \( \Rightarrow \) spherical
- \( E \) is cylinder-like \( \Rightarrow \) cylindrical

\[
0 \leq x \leq 6 \\
0 \leq y \leq \sqrt{36-x^2} \\
0 \leq z \leq \sqrt{72-x^2-y^2}
\]

\( z \): bounded below below by \( z = \sqrt{x^2+y^2} \) cone.

\( z \): bounded above by \( z = \sqrt{72-x^2-y^2} \) upper hemisphere radius \( \sqrt{72} \).
part of an ice cream cone

bounds for $\theta$: $0 \leq \theta \leq \frac{\pi}{2}$

sphere $x^2 + y^2 + z^2 = 72$

outside of cone $z = \sqrt{x^2 + y^2}$

$0 \leq \phi \leq \frac{\pi}{4}$

$0 \leq p \leq \sqrt{72}$

$z = y$

slope $= 1 \Rightarrow$ cuts QI in half

$\phi = \frac{\pi}{4}$
\[
\int_{0}^{b} \int_{0}^{\sqrt{36-x^2}} \int_{0}^{\sqrt{72-x^2-y^2}} \frac{dz \, dy \, dx}{\sqrt{x^2+y^2}} \quad dv
\]

\[
= \int_{0}^{\pi/4} \int_{0}^{\pi/2} \int_{0}^{\sqrt{72}} \rho^2 \sin \phi \, dp \, d\phi \, d\theta \, dx
\]

*outer-most, except in strange situations*

*always inner-most*
Example

\[ \iiint_E y^2 \, dv \]

\( E \): bounded by \( x^2 + y^2 + z^2 = 9 \) and \( y = 0, \ y \geq 0 \)

sphere \( \rightarrow \) spherical Gnd!

right half of sphere radius 3

\[ 0 \leq \theta \leq \pi \]
\[ 0 \leq \phi \leq \pi \]
\[ 0 \leq \rho \leq 3 \]

\[ \int_0^\pi \int_0^\pi \int_0^3 \rho^2 \sin^2 \phi \sin^2 \theta \cdot \rho^2 \sin \phi \ d\rho d\phi d\theta \]

\[ y^2 \]

\[ dv \]