12.5 Lines and Planes (continued)

lines through \((x_0, y_0, z_0)\) with direction vector \(\vec{d} = \langle a, b, c \rangle\)

is \(\vec{r} = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle\)

plane through \((x_0, y_0, z_0)\) with vector orthogonal to the plane \(\vec{n} = \langle a, b, c \rangle\)

\[a(x - x_0) + b(y - y_0) + c(z - z_0) = 0\]
Example: Equation of a line through \((5, 3, 4)\) and is perpendicular to the plane \(x - y + 2z = 8\).

\[ \vec{n} = \langle 1, -1, 2 \rangle \]

\[ x - y + 2z = 8 \]

\((5, 3, 4)\) line \(L\)

line direction \(\vec{v} = \langle 1, -1, 2 \rangle \) or \(\langle -1, 1, -2 \rangle\)

Equation of line \(L\): \( \vec{r} = \langle 5 + t, 3 - t, 4 + 2t \rangle \)

\[ x = 5 + t, \ y = 3 - t, \ z = 4 + 2t \]
at what point does this line intersect the plane?

Point?

the same point is on the line and the plane

line: \( \vec{r} = (5+t, 3-t, 4+2t) \)

plane: \( x - y + 2z = 8 \)

\[ 5+t - (3-t) + 2(4+2t) = 8 \]

\[ 6t = -2 \quad \text{intersect at } t = -\frac{1}{3} \]

from line e.g.: point is \( \left( \frac{14}{3}, \frac{10}{3}, \frac{10}{3} \right) \)
example Find equation of the line of intersection of the planes $x + y + z = 2$ and $x + 7y + 7z = 2$

![Diagram of planes and line](image)

- Side view: $\hat{n}_1$
- Plane 1
- Plane 2
- Vector $\vec{v}$ for the line is $\hat{n}_1 \times \hat{n}_2$ or $\hat{n}_2 \times \hat{n}_1$

$\Rightarrow \hat{n}_1 \times \hat{n}_2$
\[ \vec{V} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 7 & 7 \end{vmatrix} = \langle 0, -6, 6 \rangle \]

need one point on \( L \)

one way: set the two plane equations equal to each other

another way: find out where the line intersects the \( xy-, yz-, \) or \( xz- \) plane

here, \( z = 0 \)
planes: \[ x + y + z = 2 \quad x + 7y + 7z = 2 \]

at intersection w/ xy-plane, \( z = 0 \)

\[
\begin{align*}
  x + y &= 2 \quad \rightarrow \quad y = 2 - x \\
  x + 7y &= 2
\end{align*}
\]

\[
x + 7(2-x) = 2
\]

\[
-6x = -12 \quad \therefore \quad x = 2
\]

so \( y = 0 \) and \( z = 0 \)

so one pt on \( L \) is \( (2, 0, 0) \)

\[
\vec{v} = \langle 0, -6, 6 \rangle
\]

so eq is \( \vec{r} = \langle 2, -6t, 6t \rangle \)
mathematically, a **cylinder** is a surface that contains all lines parallel to some given line.

- **Circular cylinder**
  - contains all lines parallel to symmetry axis

- **Parabolic cylinder**
in 2D ($\mathbb{R}^2$)  \( y = x^2 \) is a parabola

in 3D ($\mathbb{R}^3$)  \( y = x^2 \) means for any value of \( z \), the cross-section is always \( y = x^2 \) (parabola)

\[ z = 1 \]
\[ z = 0 \]
\[ z = -1 \]

stack them
what shape is $y^2 + z^2 = 1$?
Example: \[ x^2 + y^2 + 4z^2 = 16 \]

no missing variable \( \rightarrow \) Not a cylinder

Look at the traces (hold one of variables constant at a time)

Let \( z = k \), then cross-section at \( z = k \) is

\[ x^2 + y^2 = 16 - 4k^2 \]

Circle center at \( x=0, y=0, z=k \)

\( -2 \leq k \leq 2 \)

Let \( k = 0 \) and \( k = \pm 2 \)
Now we let \( y = k \), the cross-section for each \( k \) is

\[ x^2 + 4z^2 = 16 - k^2 \]

Ellipses centered at \((0, k, 0)\) with major axis along \( x \)-axis

(y into page)

Finally, let \( x = k \), cross-sections are

\[ y^2 + 4z^2 = 16 - k^2 \]

Ellipses (same as above) major axis along \( y \)-axis

(x out of page)
put all this together

top view: circles w/ varying radii
(x vs y)

side view 1: ellipses w/ varying major and minor axes
(x vs z)

side view 2: same (same as above)
(y vs z)

circles with radius $16 - z^2$

ellipses

ellipses

think of a flattened sphere (like a piece of mentos candy)
this is called an ellipsoid
(better pic next page)
so far we have looked at cylinders and ellipsoids

there are lots more to see

there will be many names you don't recognize on HW

(but only cylinders and ellipsoids will show up)