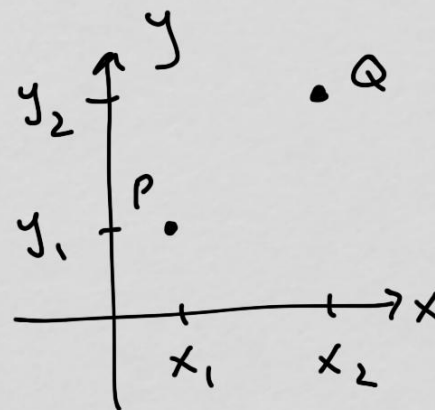


13.1 - 13.4 Review of Vectors

basic operations

$$P(x_1, y_1), Q(x_2, y_2)$$



the vector from P to Q

$$\text{is } \vec{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j}$$

from Q to P

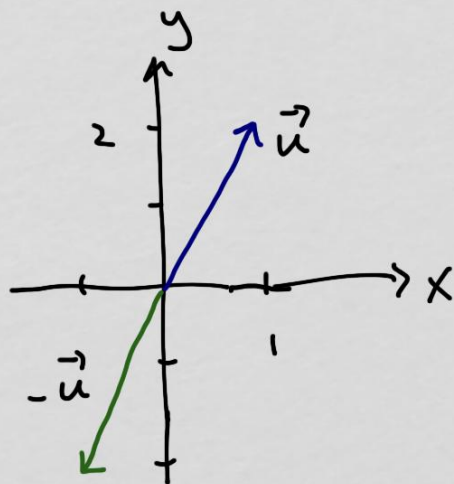
$$\text{is } \vec{QP} = \langle x_1 - x_2, y_1 - y_2 \rangle = (x_1 - x_2)\vec{i} + (y_1 - y_2)\vec{j}$$

$$\text{notice } \vec{PQ} = -\vec{QP}$$

in general, changing the sign reverses the direction

for example, $\vec{u} = \langle 1, 2 \rangle = 1\vec{i} + 2\vec{j}$

$$-\vec{u} = \langle -1, -2 \rangle = -\vec{i} - 2\vec{j}$$



the magnitude or length of a vector

$$\vec{u} = \langle x, y \rangle = x\vec{i} + y\vec{j}$$

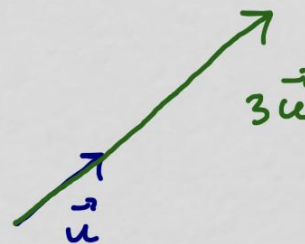
then its magnitude is $|\vec{u}| = \sqrt{x^2 + y^2}$

example: $\vec{u} = \langle 1, 2 \rangle$, then $|\vec{u}| = \sqrt{1^2 + 2^2} = \sqrt{5}$

$$\vec{v} = \langle 1, 2, 3 \rangle, \quad |\vec{v}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$\vec{u} = \langle 1, 2, 3 \rangle, \quad \vec{v} = \langle 4, 5, 6 \rangle$$

$$3\vec{u} = \langle 3 \cdot 1, 3 \cdot 2, 3 \cdot 3 \rangle = \langle 3, 6, 9 \rangle$$



$$2\vec{u} - 3\vec{v} = \underbrace{\langle 2 \cdot 1, 2 \cdot 2, 2 \cdot 3 \rangle}_{2\vec{u}} - \underbrace{\langle 3 \cdot 4, 3 \cdot 5, 3 \cdot 6 \rangle}_{3\vec{v}}$$

$$= \langle 2, 4, 6 \rangle - \langle 12, 15, 18 \rangle = \langle -10, -11, -12 \rangle$$

a unit vector is a vector whose magnitude is 1

$\vec{u} = \langle 1, 2, 3 \rangle$ is NOT a unit vector because $|\vec{u}| = \sqrt{14} \neq 1$

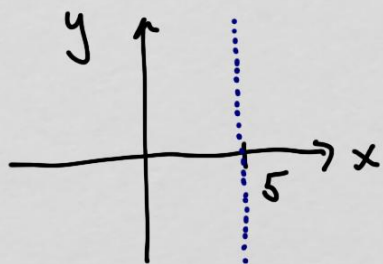
but $\frac{\vec{u}}{|\vec{u}|} = \frac{\langle 1, 2, 3 \rangle}{\sqrt{14}} = \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle$ is

a vector of length 5 in the opposite direction of $\vec{u}$
is $-5 \frac{\vec{u}}{|\vec{u}|}$

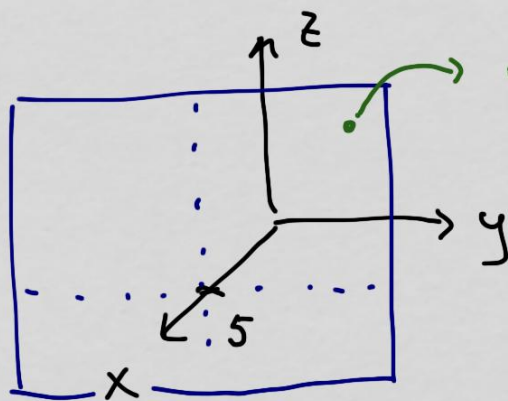
most shapes in 3D ($\mathbb{R}^3$) is very similar to their 2D ($\mathbb{R}^2$) counterparts.

for example, $x=5$ in $\mathbb{R}^2$ is a line

→ $x=5$ but y can be anything



$x=5$ in $\mathbb{R}^3$ is a plane → $x=5$ but y and z can be anything



every possible point of the form $(5, y, z)$

$(x-h)^2 + (y-k)^2 = r^2$ in $\mathbb{R}^2$ is a circle w/ center (h, k)
and radius r

$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$ in $\mathbb{R}^3$ is a sphere w/ center
 (h, k, l) and radius r

$$(x-7)^2 + (y+6)^2 + z^2 = 10$$

center: $(7, -6, 0)$ radius: $\sqrt{10}$

Standard form

non-standard to standard : complete the square

example

$$x^2 + y^2 + z^2 - 14x + 12y + 25 = 0$$

$$x^2 \textcircled{-14}x + y^2 + 12y + z^2 = -25$$

add square of $\frac{1}{2}$ of this

$$x^2 - 14x + \underline{(-7)^2} + y^2 + 12y + \underline{(6)^2} + z^2 = -25 + (-7)^2 + (6)^2$$

$$\underline{x^2 - 14x + 49} + \underline{y^2 + 12y + 36} + z^2 = -25 + 49 + 36 = 60$$

$$(x-7)^2 + (y+6)^2 + (z-0)^2 = 60$$

center: $(7, -6, 0)$

radius: $\sqrt{60}$

Dot Product

$$\vec{u} = \langle 1, 2, 3 \rangle \quad \vec{v} = \langle 4, 5, 6 \rangle$$

$$\vec{u} \cdot \vec{v} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 4 + 10 + 18 = 32$$

Dot product gives
a scalar

also, $\vec{v} \cdot \vec{u}$

the other definition: $\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta$

if $\vec{u} \perp \vec{v}$ (perpendicular or orthogonal)

then $\vec{u} \cdot \vec{v} = 0$

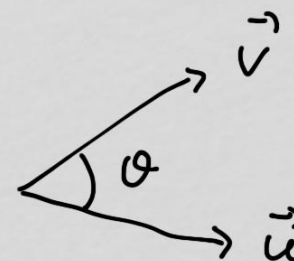
find θ between $\vec{u} = \langle 1, 2, 3 \rangle$ and $\vec{v} = \langle 4, 5, 6 \rangle$

$$\vec{u} \cdot \vec{v} = 32 = |\vec{u}| |\vec{v}| \cos \theta$$

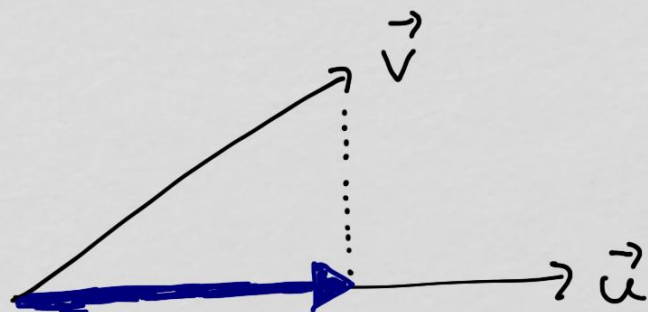
$$32 = \sqrt{14} \sqrt{77} \cos \theta$$

$$\cos \theta = \frac{32}{\sqrt{14} \sqrt{77}}$$

$$\theta \approx 0.23 \text{ rad or } 13^\circ$$



scalar and vector projections

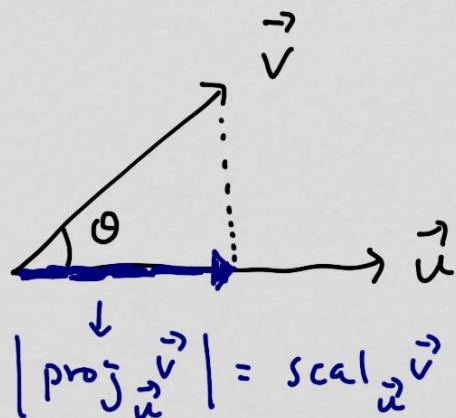


↳ this vector is the vector projection of $\vec{v}$ onto $\vec{u}$

written as $\text{proj}_{\vec{u}} \vec{v}$

its magnitude is the scalar projection of $\vec{v}$ onto $\vec{u}$

written as $\text{scal}_{\vec{u}} \vec{v}$



from trig, $\cos \theta = \frac{\text{adj}}{\text{hyp}}$

$$\cos \theta = \frac{\text{scal}_{\vec{u}} \vec{v}}{|\vec{v}|}$$

we also know: $\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta$

$$\text{so, } \cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|}$$

$$\text{so, we get: } \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} = \frac{\text{scal}_{\vec{u}} \vec{v}}{|\vec{v}|} \Rightarrow$$

$$\boxed{\text{scal}_{\vec{u}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|}}$$

$\text{proj}_{\vec{u}} \vec{v}$ is easy once we know $\text{scal}_{\vec{u}} \vec{v}$

it's just a vector in the direction of $\vec{u}$ w/ magnitude $\text{scal}_{\vec{u}} \vec{v}$

$$\rightarrow \text{proj}_{\vec{u}} \vec{v} = \text{scal}_{\vec{u}} \vec{v} \underbrace{\frac{\vec{u}}{|\vec{u}|}}$$

unit vector in direction of $\vec{u}$

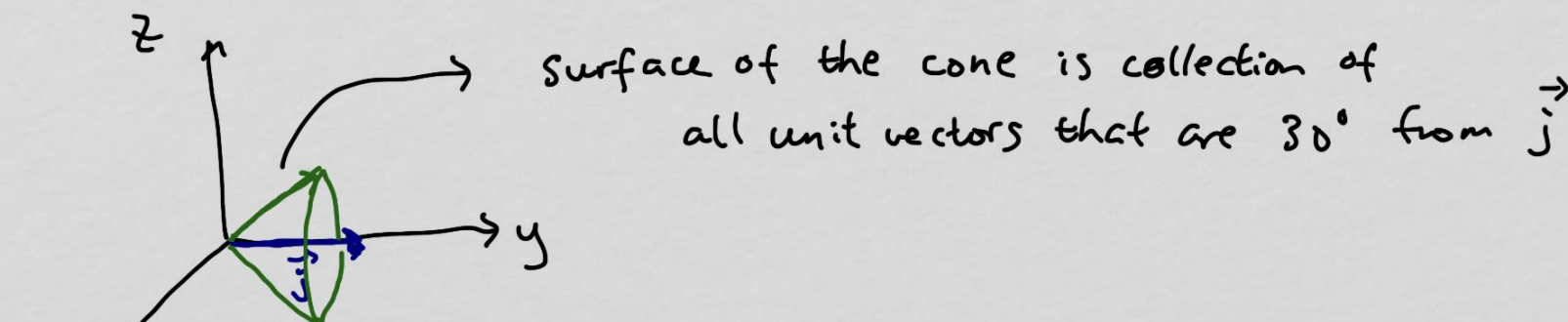
=

so doesn't change the magnitude
just gives direction

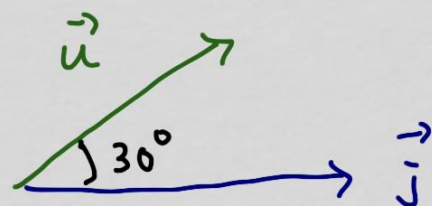
$$\boxed{\text{proj}_{\vec{u}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|} \frac{\vec{u}}{|\vec{u}|} = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|^2} \vec{u}}$$

example Let $\vec{u}$ be all possible unit vectors that form an angle of 30° with $\vec{j}$.

Find $\text{scal}_{\vec{j}} \vec{u}$.

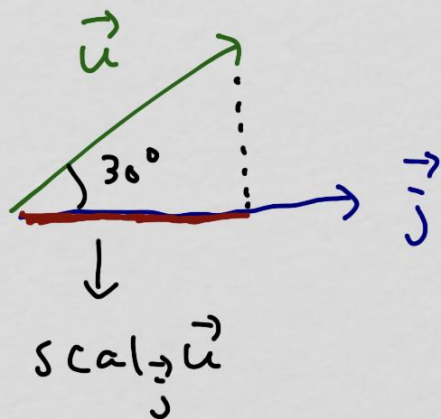


take a slice of the cone, for example, through yz -plane



($\vec{i}$ comes out screen)

this picture is the same through xy -plane, or any plane through y -axis



from trig, $\cos 30^\circ = \frac{\text{scal}_{\vec{j}} \vec{u}}{|\vec{u}|}$

so, $\text{scal}_{\vec{j}} \vec{u} = \cos 30^\circ \cdot \underbrace{|\vec{u}|}_1$ because $\vec{u}$ is unit vector

so, $\boxed{\text{scal}_{\vec{j}} \vec{u} = \frac{\sqrt{3}}{2}}$

or, $\text{scal}_{\vec{j}} \vec{u} = \frac{\vec{u} \cdot \vec{j}}{|\vec{j}|} = \frac{|\vec{u}| |\vec{j}| \cos 30^\circ}{|\vec{j}|} = \cos 30^\circ = \frac{\sqrt{3}}{2}$

Cross Product

$$\vec{u} = \langle 2, 1, 1 \rangle \quad \vec{v} = \langle 5, 0, 1 \rangle$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 5 & 0 & 1 \end{vmatrix}$$

↑
components
go into row 2

$$= \vec{i} \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 1 \\ 5 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & 1 \\ 5 & 0 \end{vmatrix}$$

determinant of sub-matrix formed by
crossing out the column and row that
 $\vec{i}$ occupies

$$= \vec{i} \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 1 \\ 5 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & 1 \\ 5 & 0 \end{vmatrix}$$

$$= \vec{i} (1 \cdot 1 - 0 \cdot 1) - \vec{j} (2 \cdot 1 - 5 \cdot 1) + \vec{k} (2 \cdot 0 - 5 \cdot 1)$$

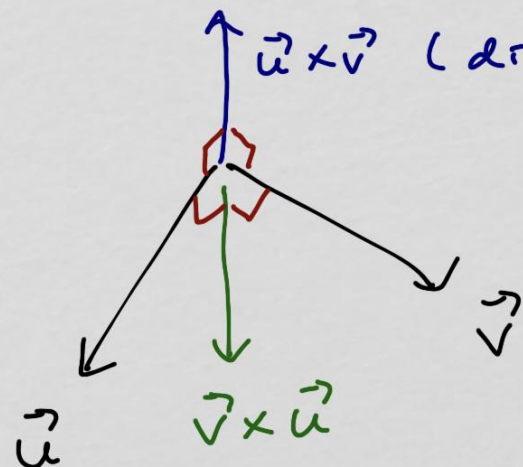
$\underbrace{\quad\quad\quad}_{\text{product of main diagonal}} \quad \underbrace{\quad\quad\quad}_{\text{product of off diagonal}}$

$$= 1 \vec{i} + 3 \vec{j} - 5 \vec{k} = \langle 1, 3, -5 \rangle$$

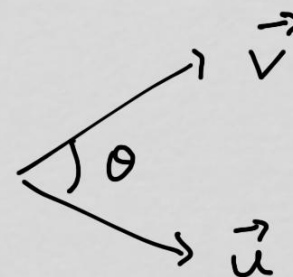
note $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v}) = \langle -1, -3, 5 \rangle$

order matters in cross product
reversing order changes sign

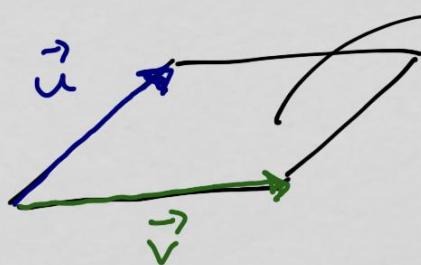
$\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$ are both perpendicular to $\vec{u}$ and $\vec{v}$



also, $|\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin \theta$



$|\vec{u} \times \vec{v}|$ is also area of a parallelogram
with $\vec{u}, \vec{v}$ as intersecting sides



$$\text{area} = |\vec{u} \times \vec{v}| = |\vec{v} \times \vec{u}|$$

magnitude is independent
of direction