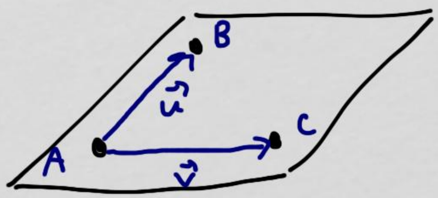
Find an equation of the plane that contains the points $\overset{A}{(1,0,-1)}$, $\overset{B}{(-5,3,2)}$, and $\overset{C}{(2,-1,4)}$.

A. $6x - 11y + z = 5$ B. $6x + 11y + z = 5$ C. $11x - 6y + z = 0$
D. $\vec{r} = 18\vec{i} - 33\vec{j} + 3\vec{k}$ E. $x - 6y - 11z = 12$

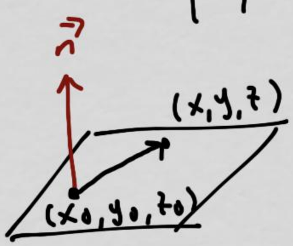


need for plane: normal vector and one point

$\vec{u} = \vec{AB} = \langle -6, 3, 3 \rangle$
$\vec{v} = \vec{AC} = \langle 1, -1, 5 \rangle$

in plane so $\vec{u} \times \vec{v}$ or $\vec{v} \times \vec{u}$ is normal to plane

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -6 & 3 & 3 \\ 1 & -1 & 5 \end{vmatrix} = \langle 18, 33, 3 \rangle = \vec{n} \text{ (normal vector)}$$



$\vec{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

here, use $x_0 = 1$, $y_0 = 0$, $z_0 = -1$

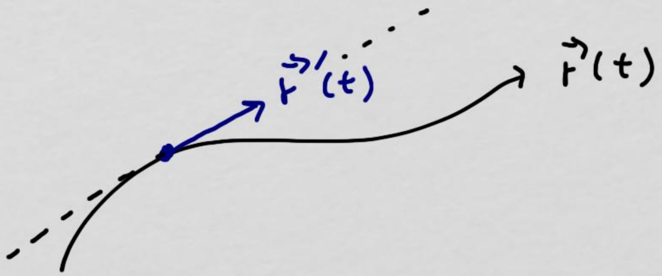
$\langle 18, 33, 3 \rangle \cdot \langle x-1, y, z+1 \rangle = 0$

$18x - 18 + 33y + 3z + 3 = 0$
$18x + 33y + 3z = 15$
$6x + 11y + z = 5$

Find parametric equations of the line tangent to the curve $\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}$ at the point $(2, 4, 8)$

A. $x = 2 + t, y = 4 + 4t, z = 8 + 12t$ B. $x = 1 + 2t, y = 4 + 4t, z = 12 + 8t$

C. $x = 2t, y = 4t, z = 8t$ D. $x = t, y = 4t, z = 12t$ E. $x = 2 + t, y = 2 + 2t, z = 8 + 3t$



line : need direction vector and one point

direction vector : $\vec{r}'(t) = \langle 1, 2t, 3t^2 \rangle$

at tangent point $(2, 4, 8)$, $t = ?$

$\vec{r}(t) = \langle t, t^2, t^3 \rangle = \langle 2, 4, 8 \rangle \longrightarrow t = 2$

$\vec{r}'(2) = \langle 1, 4, 12 \rangle$ direction vector $\vec{v}$

equation of the line : $\vec{r}(t) = \langle 2, 4, 8 \rangle + t \langle 1, 4, 12 \rangle$

$= \langle 2 + t, 4 + 4t, 8 + 12t \rangle$

$x = 2 + t$

$y = 4 + 4t$

$z = 8 + 12t$

A smooth parametrization of the semicircle which passes through the points $(1, 0, 5)$, $(0, 1, 5)$ and $(-1, 0, 5)$ is

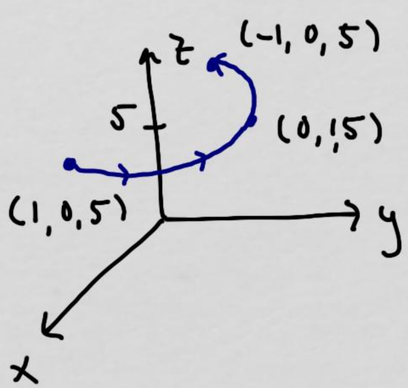
A. $\vec{r}(t) = \sin t\vec{i} + \cos t\vec{j} + 5\vec{k}, 0 \le t \le \pi$

B. $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + 5\vec{k}, 0 \le t \le \pi$

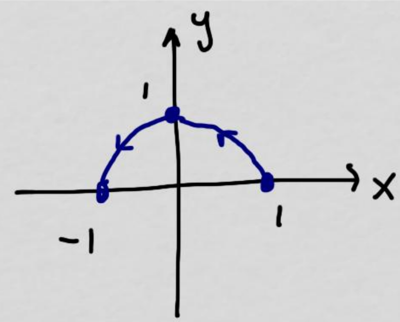
C. $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + 5\vec{k}, \frac{\pi}{2} \le t \le \frac{3\pi}{2}$

D. $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + 5\vec{k}, 0 \le t \le \frac{\pi}{2}$

E. $\vec{r}(t) = \sin t + \cos t\vec{j} + 5\vec{k}, \frac{\pi}{2} \le t \le \frac{3\pi}{2}$



top view



circle radius 1
from $\theta = 0$ to $\theta = \pi$
all at $z = 5$

$\vec{r}(t) = \langle 1 \cdot \cos t, 1 \cdot \sin t, 5 \rangle$

$= \langle \cos t, \sin t, 5 \rangle \quad 0 \le t \le \pi$

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The length of the curve $\vec{r}(t) = \frac{2}{3}(1+t)^{\frac{3}{2}}\vec{i} + \frac{2}{3}(1-t)^{\frac{3}{2}}\vec{j} + t\vec{k}$, $-1 \le t \le 1$ is

A. $\sqrt{3}$ B. $\sqrt{2}$ C. $\frac{1}{2}\sqrt{3}$ D. $2\sqrt{3}$ E. $\sqrt{2}$

length : $\int_a^b |\vec{r}\,'|\,dt$ $\vec{r}\,'(t) = \langle (1+t)^{1/2}, -(1-t)^{1/2}, 1 \rangle$, $-1 \le t \le 1$

$|\vec{r}\,'| = \sqrt{(1+t) + (1-t) + 1} = \sqrt{3}$

$L = \int_{-1}^{1} \sqrt{3}\,dt = \sqrt{3}\,t \Big|_{-1}^{1} = \sqrt{3}\cdot 2 = 2\sqrt{3}$

Match the graphs of the equations with their names:

(1) $x^2 + y^2 + z^2 = 4$ (a) paraboloid

(2) $x^2 + z^2 = 4$ (b) sphere

(3) $x^2 + y^2 = z^2$ (c) cylinder

(4) $x^2 + y^2 = z$ (d) double cone

(5) $x^2 + 2y^2 + 3z^2 = 1$ (e) ellipsoid

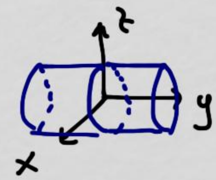
A. 1b, 2c, 3d, 4a, 5e B. 1b, 2c, 3a, 4d, 5e C. 1e, 2c, 3d, 4a, 5b

D. 1b, 2d, 3a, 4c, 5e E. 1d, 2a, 3b, 4e, 5c

(1) sphere of radius 2 – (b)

(2) $x^2 + z^2 = 4$ missing one variable → cylinder – (c)

xz-slices are circles of radius 2

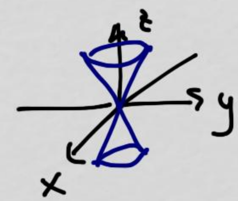


(3) $x^2 + y^2 = z^2$ xz-trace: $z^2 = x^2$ $z = \pm x$

yz-trace: $z^2 = y^2$ $z = \pm y$

xy-slices are circles that grow as $z \to \pm\infty$

double cone – (d)

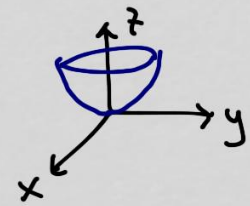


(4) $x^2 + y^2 = z$ xz-trace: $z = x^2$ (parabola)

yz-trace: $z = y^2$ (parabola)

xy-traces are circles for $z \geq 0$

paraboloid – (a)



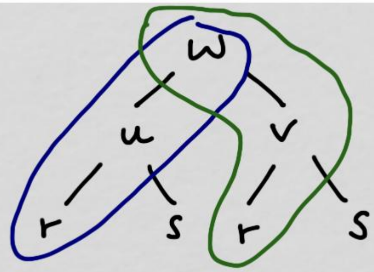
(5) looks likes sphere but different coefficients – ellipsoid

If $w = e^{uv}$ and $u = r + s$, $v = rs$, find $\frac{\partial w}{\partial r}$.

A. $e^{(r+s)rs}(2rs + r^2)$ B. $e^{(r+s)rs}(2rs + s^2)$ C. $e^{(r+s)rs}(2rs + r^2)$

D. $e^{(r+s)rs}(1 + s)$ E. $e^{(r+s)rs}(r + s^2)$.

Chain rule draw tree :



$\frac{\partial w}{\partial r}$: go down branches w r at bottom.

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial r}$$

$$= (e^{uv} \cdot v)(1) + (e^{uv} \cdot u)(s)$$ write everything in terms of r, s

$$= rs\,e^{(r+s)rs} + (r+s)e^{(r+s)rs} \cdot s$$

$$= e^{(r+s)rs}(rs + sr + s^2) = e^{(r+s)rs}(2rs + s^2)$$

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Assuming that the equation $xy^2 + 3z = \cos(z^2)$ defines z implicitly as a function of x and y, find $\frac{\partial z}{\partial x}$.

A. $\frac{y^2}{3-\sin(z^2)}$ B. $\frac{-y^2}{3+\sin(z^2)}$ C. $\frac{y^2}{3+2z\sin(z^2)}$ D. $\frac{-y^2}{3+2z\sin(z^2)}$ E. $\frac{-y^2}{3-2z\sin(z^2)}$

z is an unknown function of x, y $\rightarrow$ $z = z(x,y)$

x and y are independent of each other

$$xy^2 + 3z = \cos(z^2)$$

take $\frac{\partial}{\partial x}$ on both sides: $\frac{\partial}{\partial x}(xy^2 + 3z) = \frac{\partial}{\partial x}\cos(z^2)$

$$y^2 + 3\frac{\partial z}{\partial x} = -\sin(z^2)\cdot 2z\frac{\partial z}{\partial x}$$

solve for $\frac{\partial z}{\partial x}$: $2z\sin(z^2)\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial x} = -y^2$

$$\frac{\partial z}{\partial x}\left(2z\sin(z^2) + 3\right) = -y^2$$

$$\frac{\partial z}{\partial x} = \frac{-y^2}{2z\sin(z^2)+3}$$

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For the function $f(x,y) = x^2y$, find a unit vector $\vec{u}$ for which the directional derivative $D_{\vec{u}}f(2,3)$ is zero.

A. $\vec{i} + 3\vec{j}$ B. $\frac{\vec{i}+3\vec{j}}{\sqrt{10}}$ C. $\vec{i} - 3\vec{j}$ D. $\frac{\vec{i}-3\vec{j}}{\sqrt{10}}$ E. $\frac{3\vec{i}-\vec{j}}{\sqrt{10}}$.

directional derivative : $D_{\vec{u}} f(x,y) = \vec{\nabla} f(x,y) \cdot \vec{u}$ unit vector

let $\vec{u} = \langle a, b \rangle$ → since $\vec{u}$ is unit vector, $a^2 + b^2 = 1$

$\vec{\nabla} f = \langle 2xy, x^2 \rangle$ $\vec{\nabla} f(2,3) = \langle 12, 4 \rangle$ sub

$D_{\vec{u}} f(2,3) = 0 = \langle 12, 4 \rangle \cdot \langle a, b \rangle = 12a + 4b$ $b = -3a$

$a^2 + (-3a)^2 = 1$

$10a^2 = 1$

$a^2 = \frac{1}{10}$ $a = \frac{1}{\sqrt{10}}$ and $b = -3a = -\frac{3}{\sqrt{10}}$

$\vec{u} = \langle a, b \rangle = \langle \frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}} \rangle$

Find an equation of the tangent plane to the surface $x^2 + 2y^2 + 3z^2 = 6$ at the point $(1, 1, -1)$.

A. $-x + 2y + 3z = 2$ B. $2x + 4y - 6z = 6$ C. $x - 2y + 3z = -4$

D. $2x + 4y - 6z = 0$ E. $x + 2y - 3z = 6$.

two ways : use the formula $z - z_0 = f_x (x - x_0) + f_y (y - y_0)$

$$x^2 + 2y^2 + 3z^2 = 6$$

$$z^2 = 2 - \frac{1}{3}x^2 - \frac{2}{3}y^2$$

$$z = f(x,y) = -\sqrt{2 - \frac{1}{3}x^2 - \frac{2}{3}y^2}$$

needs to be negative because at $x = 1$, $y = 1$, $z = -1$

$$f_x = -\frac{1}{2}\left(2 - \frac{1}{3}x^2 - \frac{2}{3}y^2\right)^{-1/2}\left(-\frac{2}{3}x\right) = \frac{x}{3\sqrt{2 - \frac{1}{3}x^2 - \frac{2}{3}y^2}}$$

$$f_y = -\frac{1}{2}\left(2 - \frac{1}{3}x^2 - \frac{2}{3}y^2\right)^{-1/2}\left(-\frac{4}{3}y\right) = \frac{2y}{3\sqrt{2 - \frac{1}{3}x^2 - \frac{2}{3}y^2}}$$

at $(1, 1, -1)$ $f_x = \frac{1}{3}$, $f_y = \frac{2}{3}$

Sub into $z - z_0 = f_x (x - x_0) + f_y (y - y_0)$

$$z - (-1) = \frac{1}{3}(x-1) + \frac{2}{3}(y-1)$$

$$z + 1 = \frac{1}{3}(x-1) + \frac{2}{3}(y-1)$$

$$3(z+1) = (x-1) + 2(y-1)$$

$$3z + 3 = x - 1 + 2y - 2$$

$$x + 2y - 3z = 6$$

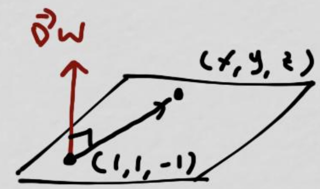
---

second way: treat $x^2 + 2y^2 + 3z^2 = 6$ as a level surface of the function $x^2 + 2y^2 + 3z^2 = w(x, y, z)$

then, $\vec{\nabla} w$ at $(1, 1, -1)$ is normal to the surface, so with it and the point $(1, 1, -1)$ we can find the tangent plane

$$w = x^2 + 2y^2 + 3z^2$$

$$\vec{\nabla} w = \langle 2x, 4y, 6z \rangle$$



$$\vec{\nabla} w\,(1,1,-1) = \langle 2, 4, -6 \rangle$$

the point on the tangent plane is $(1, 1, -1)$

$$\vec{\nabla} w \cdot \langle x-1,\, y-1,\, z+1 \rangle = 0$$

$$\langle 2, 4, -6 \rangle \cdot \langle x-1,\, y-1,\, z+1 \rangle = 0$$

$$2(x-1) + 4(y-1) - 6(z+1) = 0$$

$$2x + 4y - 6z = 12$$

$$x + 2y - 3z = 6$$

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Suppose that the Celsius temperature at the point (x,y,z) on the sphere $x^2 + y^2 + z^2 = 1$ is $T = 200xyz^2$. Locate the highest and lowest temperatures on the sphere.

The highest temperature on the sphere is ☐ °C. (Simplify your answer.)

The lowest temperature on the sphere is ☐ °C. (Simplify your answer.)

equivalent to: find max/min of $f = 200xyz^2$ subject to constraint

$$g = x^2+y^2+z^2-1 = 0$$

solve $\vec{\nabla} f = \lambda \vec{\nabla} g$

$$\langle 200yz^2, 200xz^2, 400xyz \rangle = \lambda \cdot \langle 2x, 2y, 2z \rangle$$

$$200yz^2 = \lambda \cdot 2x \longrightarrow \lambda = \frac{100yz^2}{x}$$

$$200xz^2 = \lambda \cdot 2y \longrightarrow \lambda = \frac{100xz^2}{y}$$

$$400xyz = \lambda \cdot 2z \longrightarrow \lambda = 200xy$$

so, $\dfrac{100yz^2}{x} = \dfrac{100xz^2}{y} = 200xy$

multiply all by xy

$$100y^2z^2 = 100x^2z^2 = 200x^2y^2$$

$x^2 = y^2$ $\qquad$ $z^2 = 2y^2 = 2x^2$ because $x^2 = y^2$

Sub into g: $\quad x^2 + y^2 + z^2 = 1$

$$x^2 + x^2 + 2x^2 = 1 \qquad x^2 = \frac{1}{4} \qquad x = \pm\frac{1}{2}$$

$$y^2 = x^2 \qquad y = \pm\frac{1}{2}$$

$$z^2 = 2x^2 = \frac{1}{2} \qquad z = \pm\frac{1}{\sqrt{2}}$$

$$T = 200xyz^2$$

max when x, y have same sign: $T_{max} = 200 \cdot \frac{1}{4} \cdot \frac{1}{2} = 25$

min when x, y have opposite signs: $T_{min} = -200 \cdot \frac{1}{4} \cdot \frac{1}{2} = -25$