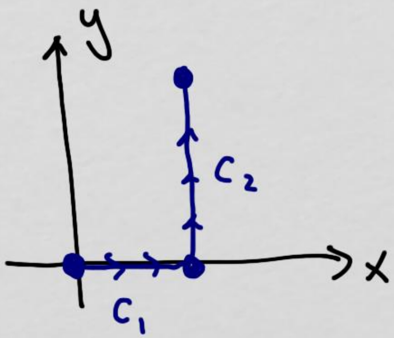
Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = y\vec{i} + x^2\vec{j}$ and C is composed of the line segments from $(0,0)$ to $(1,0)$ and from $(1,0)$ to $(1,2)$.

A. 0 B. $\frac{2}{3}$ C. $\frac{5}{6}$ D. 2 E. 3



$C_1: \vec{r}(t) = \langle t, 0 \rangle \quad 0 \le t \le 1 \quad \vec{r}\,' = \langle 1, 0 \rangle$

$C_2: \vec{r}(t) = \langle 1, t \rangle \quad 0 \le t \le 2 \quad \vec{r}\,' = \langle 0, 1 \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$= \int_0^1 \langle 0, t^2 \rangle \cdot \langle 1, 0 \rangle \, dt + \int_0^2 \langle t, 1 \rangle \cdot \langle 0, 1 \rangle \, dt$$

$$= \int_0^1 0 \, dt + \int_0^2 1 \, dt = t \Big|_0^2 = 2$$

Are the following statements true or false?

1. The line integral $\int_C (x^3+2xy)dx + (x^2-y^2)dy$ is independent of path in the xy-plane. T
2. $\int_C (x^3+2xy)dx + (x^2-y^2)dy = 0$ for every closed oriented curve C in the xy-plane. T
3. There is a function $f(x,y)$ defined in the xy-plane, such that grad $f(x,y) = (x^3+2xy)\vec{i} + (x^2-y^2)\vec{j}$. T

A. all three are false
B. 1 and 2 are false, 3 is true
C. 1 and 2 are true, 3 is false
D. 1 is true, 2 and 3 are false
E. all three are true

1. $\int_C (x^3+2xy)dx + (x^2-y^2)dy = \int_C \underbrace{\langle x^3+2xy, x^2-y^2\rangle}_{\vec{F}} \cdot \underbrace{\langle dx, dy\rangle}_{d\vec{r}}$

$\int_C \vec{F}\cdot d\vec{r}$ is independent of path if $\vec{F}$ is conservative

↳ for 2D $\vec{F} = \langle f, g\rangle$
$g_x = f_y$

here, $f = x^3+2xy$ $\quad g = x^2-y^2$

$f_y = 2x$ $\quad g_x = 2x$ $\quad f_y = g_x$ so $\vec{F}$ is conservative

so $\int_C \vec{F}\cdot d\vec{r}$ is independent of path

2. $\int_C \vec{F}\cdot d\vec{r} = 0$ over a closed path if $\vec{F}$ is conservative

because of the Fundamental Theorem of Line Integrals: $\int_C \vec{F}\cdot d\vec{r} = \phi(B) - \phi(A)$
where $\vec{\nabla}\phi$

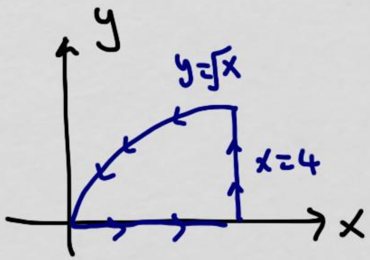
and if C is closed then $B = A$

we know from 1. that $\vec{F}$ is conservative, so 2 is also true

3. True, it's the same reason why 2 is true
($\vec{F}$ is conservative and is the gradient of some potential function).

Evaluate $\int_C y^2 dx + 6xy\, dy$ where C is the boundary curve of the region bounded by $y = \sqrt{x}$, $y = 0$ and $x = 4$, in the counterclockwise direction.

A. 0 B. 4 C. 8 D. 16 E. 32



$$\int_C y^2 dx + 6xy\, dy = \int_C \langle y^2, 6xy \rangle \cdot \langle dx, dy \rangle$$

$$= \int_C \vec{F} \cdot d\vec{r}$$

option 1: do another line integral but with 3 segments

$\longrightarrow$ option 2: $\boxed{\text{Green's Theorem}}$

$$\int_C f\, dx + g\, dy = \iint_R (g_x - f_y)\, dA$$

be careful: must have counterclockwise orientation

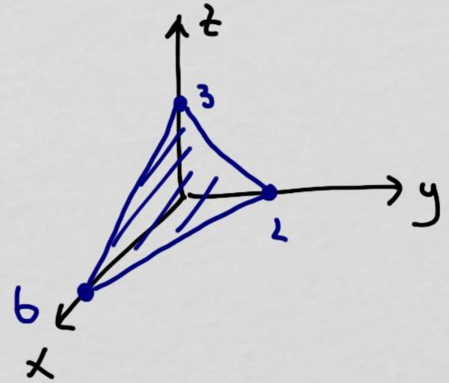
$$g_x - f_y = 6y - 2y = 4y$$

R: $0 \le x \le 4$
$0 \le y \le \sqrt{x}$

$$\int_0^4 \int_0^{\sqrt{x}} 4y\, dy\, dx = \int_0^4 2y^2 \Big|_0^{\sqrt{x}} dx = \int_0^4 2x\, dx = x^2 \Big|_0^4$$
$$= 16$$

Find the area of the portion of the plane $x + 3y + 2z = 6$ that lies in the first octant.

A. $3\sqrt{11}$ B. $6\sqrt{7}$ C. $6\sqrt{14}$ D. $3\sqrt{14}$ E. $6\sqrt{11}$.

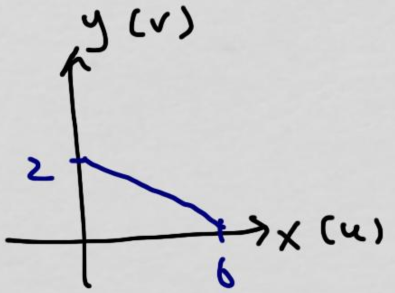


$$\boxed{\text{Surface area: } \iint_S dS = \iint_S |\vec{r}_u \times \vec{r}_v|\, dA}$$

parametrize surface

let $u = x$, $v = y$, then $z = 3 - \frac{1}{2}x - \frac{3}{2}y$

$= 3 - \frac{1}{2}u - \frac{3}{2}v$



$0 \le u \le 6$

$0 \le v \le 2 - \frac{1}{3}u$

$\vec{r}(u,v) = \langle u, v, 3 - \frac{1}{2}u - \frac{3}{2}v \rangle$

$\vec{r}_u = \langle 1, 0, -\frac{1}{2} \rangle$

$\vec{r}_v = \langle 0, 1, -\frac{3}{2} \rangle$

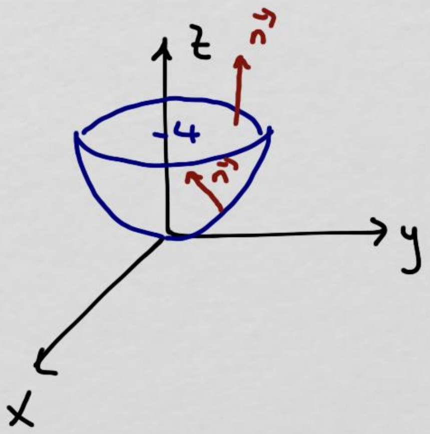
$\vec{r}_u \times \vec{r}_v = \langle \frac{1}{2}, \frac{3}{2}, 1 \rangle$ $\quad |\vec{r}_u \times \vec{r}_v| = \sqrt{\frac{1}{4} + \frac{9}{4} + 1} = \frac{\sqrt{14}}{2}$

$$\iint_S dS = \int_0^6 \int_0^{2-\frac{1}{3}u} \frac{\sqrt{14}}{2}\, dv\, du = \frac{\sqrt{14}}{2} \underbrace{\int_0^6 \int_0^{2-\frac{1}{3}u} dv\, du}_{\text{area of triangle on xy-plane}}$$

$$= \frac{\sqrt{14}}{2} \cdot \frac{1}{2} \cdot 6 \cdot 2 = 3\sqrt{14}$$

If Σ is the part of the paraboloid $z = x^2 + y^2$ with $z \le 4$, $\vec{n}$ is the unit normal vector on Σ directed upward, and $\vec{F}(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$, then $\iint_\Sigma \vec{F}\cdot\vec{n}\,dS =$

A. 0 B. 8π C. 4π D. -4π E. -8π



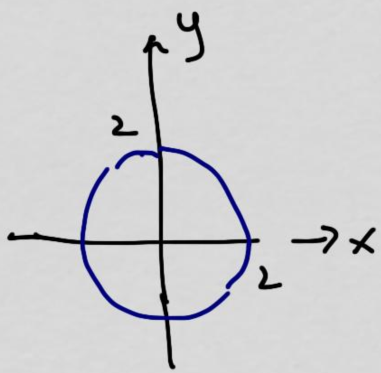
$$\iint_\Sigma \vec{F}\cdot d\vec{S} = \iint_\Sigma \vec{F}\cdot\vec{n}\,dS$$

parametrize Σ : cylindrical is good here

let $r = u$, $\theta = v$

$\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ u^2 \rangle$

↳ because $z = x^2 + y^2$
$= u^2\cos^2 v + u^2\sin^2 v$
$= u^2$



$0 \le u \le 2$

$0 \le v \le 2\pi$

$\vec{r}_u = \langle \cos v,\ \sin v,\ 2u \rangle$

$\vec{r}_v = \langle -u\sin v,\ u\cos v,\ 0 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle -2u^2\cos v,\ -2u^2\sin v,\ u \rangle$

is this upward? yes, because the z-component is positive since $0 \le u \le 2$

$$\iint_\Sigma \vec{F}\cdot\vec{n}\,dS = \int_0^{2\pi}\int_0^2 \underbrace{\langle u\cos v,\ u\sin v,\ u^2\rangle}_{\vec{F}=\langle x,y,z\rangle}\cdot\underbrace{\langle -2u^2\cos v,\ -2u^2\sin v,\ u\rangle}_{d\vec{S}}\ \boxed{du\,dv}$$

$$= \int_0^{2\pi}\int_0^2 \left(-2u^3\cos^2 v - 2u^3\sin^2 v + u^3\right)\boxed{du\,dv}$$

$$= \int_0^{2\pi}\int_0^2 \left(-2u^3+u^3\right)du\,dv = \int_0^{2\pi}\int_0^2 -u^3\,du\,dv$$

$$= \int_0^{2\pi} -\frac{1}{4}u^4\Big|_0^2\,dv = \int_0^{2\pi} -4\,dv = -8\pi$$

do not supply the extra r (u) because we started in cylindrical so $\vec{r}_u\times\vec{r}_v$ ALWAYS supply whatever is needed.

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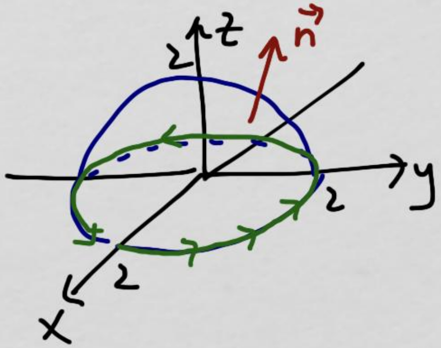
Use Stoke's theorem to evaluate $\iint_S \mathrm{curl}\vec{F}\cdot d\vec{S}$, where

$$\vec{F}(x,y,z) = x^2e^{yz}\vec{i} + y^2e^{xz}\vec{j} + z^2e^{xy}\vec{k},$$

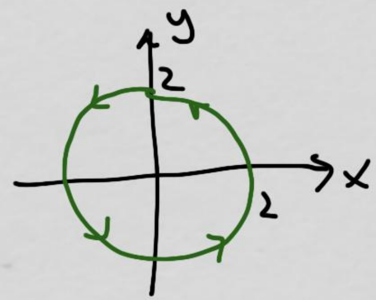
and S is the hemisphere $x^2+y^2+z^2=4$, $z\geq 0$, oriented upward.

A. $-\pi/3$ B. 2π C. 0 D. $\frac{4}{3}$ E. 2π



since $\vec{n}$ points up, by the right hand rule the boundary curve (circle on xy-plane) goes counterclockwise when viewed from above

Stokes' Theorem: $\iint_S \mathrm{curl}\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r}$



parametrize C: $\vec{r}(t) = \langle 2\cos t, 2\sin t, 0\rangle$ $0\leq t\leq 2\pi$

$\vec{F} = \langle 4\cos^2 t\, e^0, 4\sin^2 t\, e^0, 0\rangle$

$= \langle 4\cos^2 t, 4\sin^2 t, 0\rangle$

$\vec{r}\,' = \langle -2\sin t, 2\cos t, 0\rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle 4\cos^2 t,\ 4\sin^2 t,\ 0\rangle \cdot \langle -2\sin t,\ 2\cos t,\ 0\rangle\, dt$$

$$= \int_0^{2\pi} -8\cos^2 t\sin t + 8\sin^2 t\cos t \quad dt$$

$$= \int_0^{2\pi} -8\cos^2 t\sin t\, dt + \int_0^{2\pi} 8\sin^2 t\cos t\, dt$$

$u = \cos t$
$du = -\sin t\, dt$

$u = \sin t$
$du = \cos t\, dt$

$$= \int_1^1 8u^2\, du + \int_0^0 8u^2\, du = 0$$

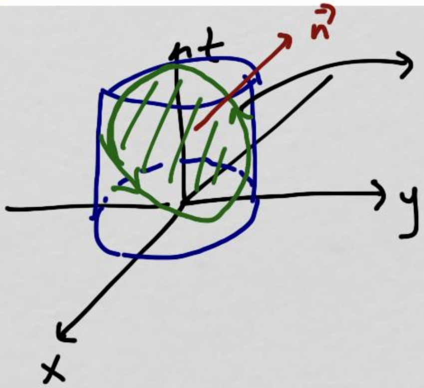
only use Stokes' if we are doing the surface integral of the curl of $\vec{F}$ AND the surface must have a boundary.

Use Stoke's theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$, where

$$\vec{F}(x, y, z) = x^2 z\vec{i} + xy^2\vec{j} + z^2\vec{k},$$

and C is the curve of intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$ oriented counterclockwise as viewed from above.

A. $\frac{81\pi}{2}$ B. $\frac{\pi}{2}$ C. 1 D. $\frac{3\pi}{8}$ E. 9π



intersection of $x+y+z=1$ and $x^2+y^2=9$
(not trivial to parametrize)

so $\int_C \vec{F} \cdot d\vec{r}$ is probably messy

therefore, Stokes' : $\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 z & xy^2 & z^2 \end{vmatrix} = \langle 0, x^2, y^2 \rangle$$

parametrize surface (slanted ellipse)
cylindrical is good for obvious reason

$u = r, \quad v = \theta$

$$\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ \underbrace{1 - u\cos v - u\sin v}_{\text{from } x+y+z=1} \rangle$$

$$0 \le u \le 3$$
$$0 \le v \le 2\pi$$

$$\vec{r}_u = \langle \cos v, \sin v, -\cos v - \sin v \rangle$$

$$\vec{r}_v = \langle -u\sin v,\ u\cos v,\ u\sin v - u\cos v \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle u, u, u \rangle \quad \text{(orientation is correct)}$$

$$\iint_S \operatorname{curl}\vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^3 \langle 0, u^2\cos^2 v, u^2\sin^2 v\rangle \cdot \langle u, u, v\rangle \, du\, dv$$

$$= \int_0^{2\pi}\int_0^3 u^3\cos^2 v + u^3\sin^2 v \, du\, dv = \int_0^{2\pi}\int_0^3 u^3 \, du\, dv = \dots = \frac{81\pi}{2}$$

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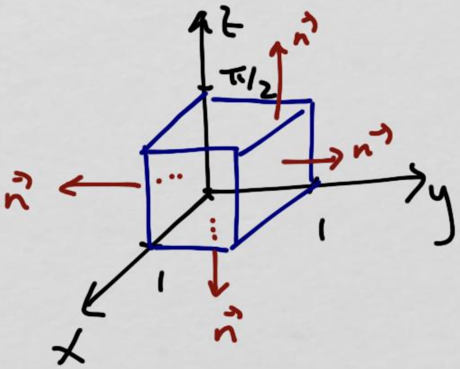
If $\vec{F}(x,y,z) = \cos z\vec{i} + \sin z\vec{j} + xy\vec{k}$, Σ is the complete boundary of the rectangular solid region bounded by the planes $x=0$, $x=1$, $y=0$, $y=1$, $z=0$ and $z=\frac{\pi}{2}$, and $\vec{n}$ is the outward unit normal on Σ, then $\iint_\Sigma \vec{F}\cdot\vec{n}\,dS =$

A. 0 B. $\frac{1}{2}$ C. 1 D. $\frac{\pi}{2}$ E. 2



Divergence Theorem : $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D \operatorname{div}\vec{F}\,dV$

(good with closed surface)

$\vec{F} = \langle \cos z, \sin z, xy \rangle$

$\operatorname{div}\vec{F} = 0 + 0 + 0 = 0$

so, $\iiint_D \operatorname{div}\vec{F}\,dV = \iiint_D 0\,dV = 0$