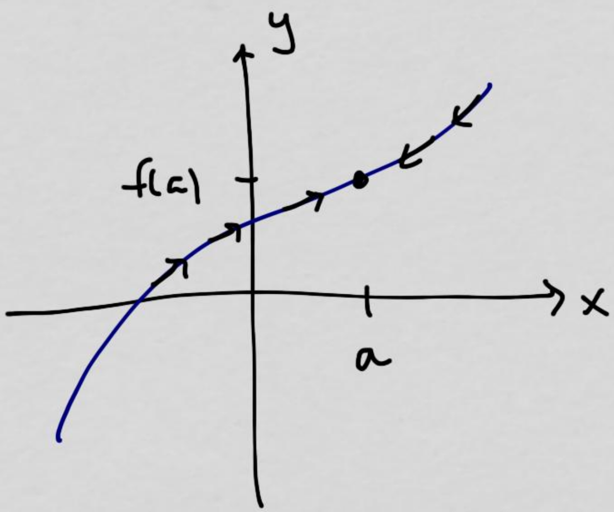
15.2 Limits and Continuity

recall that $\lim_{x \to a} f(x) = L$ that means we can make $f(x)$ as close to L as we want by making x as close to a as needed. (one-variable function)

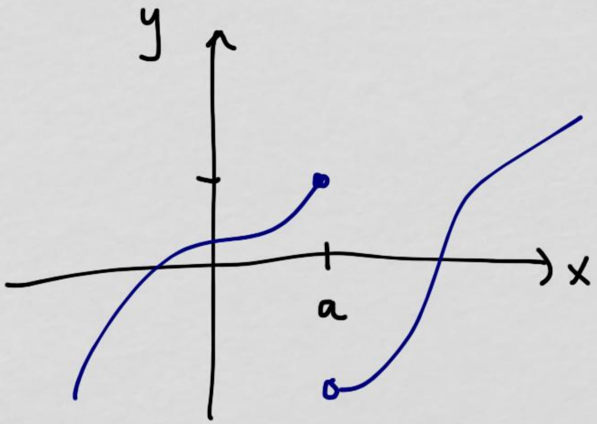
if the limit exists, then it doesn't depend on how we approach $x=a$

$$\left(\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a} f(x) = L \right)$$



$\lim_{x \to a} f(x)$ exists

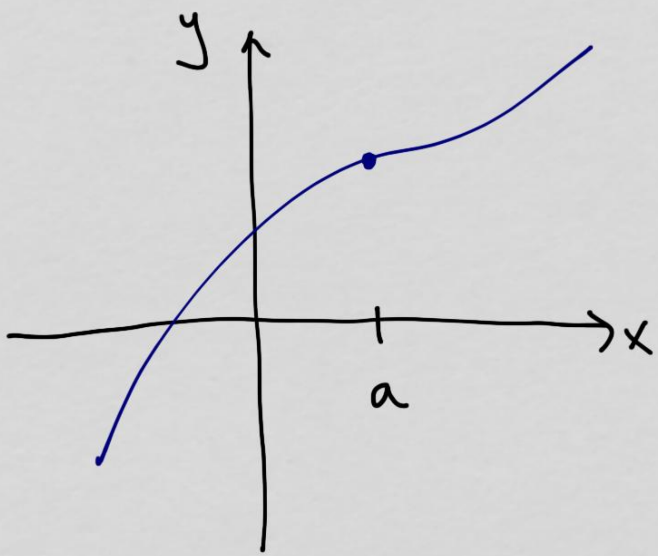
because $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$



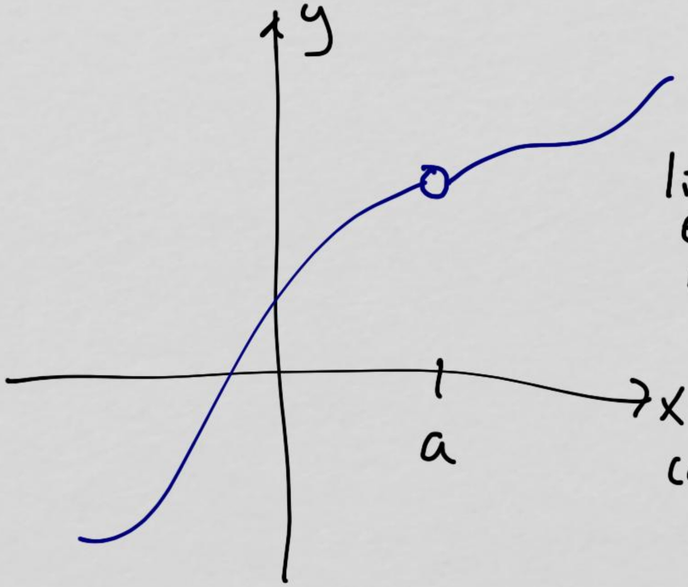
$\lim_{x\to a} f(x)$ DNE

because $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$

moreover, if $\lim_{x\to a} f(x) = f(a)$ then $f(x)$ is <u>continuous</u> at $x=a$



continuous at $x=a$



limit exist but $f(x)$ is <u>NOT</u> continuous at $x=a$

we know many types of functions are continuous

polynomials, sine, cosine, exponentials : continuous everywhere

rationals, logarithmic : continuous when defined

so, if we know a function is continuous at $x=a$, then

$\lim_{x \to a} f(x)$ is simply $f(a)$

> Almost all of the concepts from one-variable case carry over to two-variable case

$\lim_{(x,y)\to(a,b)} f(x,y) = L$ means $f(x,y)$ can be made as close to L as we want by making (x,y) as close to (a,b) as needed

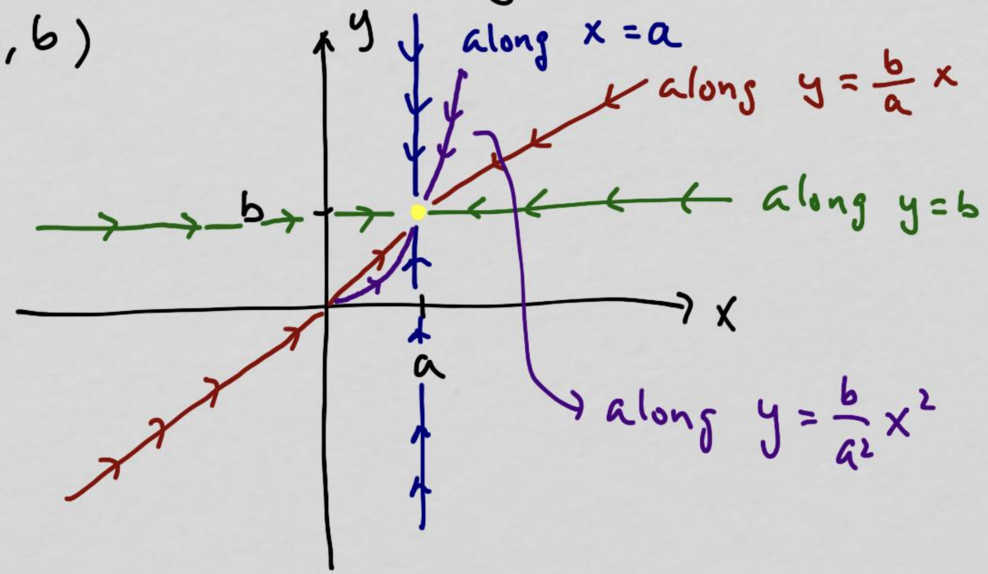
$f(x,y)$ is continuous at (a,b) if $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

now we look at how things for two-variable function are different from a one-variable function when dealing with limits

in one-variable case, there are only two way to approach $x=a$

→ from left $(x \to a^-)$ and from right $(x \to a^+)$

in two-variable case, there are infinitely-many ways for (x,y) to approach (a,b)



however, just like with one-variable case, ALL paths must lead to the same limit if $\lim_{(x,y)\to(a,b)} f(x,y)$ exists

but again, we can often take advantage of continuity to find limits.

example $$\lim_{(x,y)\to(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right)$$

but worrying about the paths, think continuity first

$\ln\left(\frac{1+y^2}{x^2+xy}\right)$ is logarithmic, so it is continuous wherever it defined

$\longrightarrow$ if defined at (a,b), then continuity implies $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

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here, we want $(x,y) \to (1,0)$

is $\ln\left(\frac{1+y^2}{x^2+xy}\right)$ defined at $(1,0)$? $\ln\left(\frac{1+0^2}{1^2+1\cdot 0}\right) = \ln(1) = 0$

yes.

therefore, $\ln\left(\frac{1+y^2}{x^2+xy}\right)$ is continuous at $(1,0)$ AND

$$\lim_{(x,y)\to(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right) = \ln\left(\frac{1+0^2}{1^2+1\cdot 0}\right) = \ln(1) = \boxed{0}$$

so, in this case, we don't need to worry about the path.

(because continuity guarantees the existence of a limit)

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<u>example</u>

$$\lim_{(x,y)\to(0,0)} \frac{y^2-4x^2}{2x^2+y^2}$$

$\dfrac{y^2-4x^2}{2x^2+y^2}$ is a rational function which is continuous wherever defined

is it defined at $(0,0)$?

no, because at $(0,0)$, we appear to get $\dfrac{0}{0}$

but the limit may still exist

but continuity won't help this time

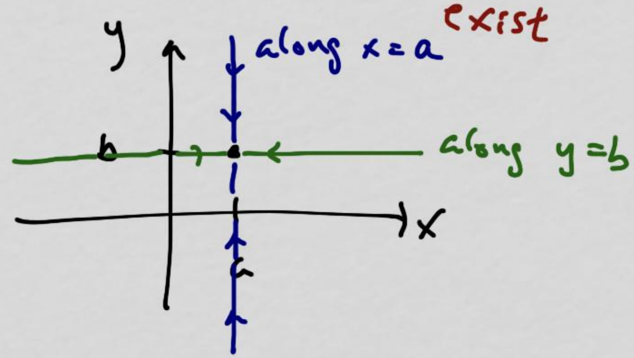
now we need worry about the paths to get to $(0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{y^2-4x^2}{2x^2+y^2}$$

a b

if the limit were to exist, then ALL paths MUST lead to the same limit

if we can find two path that lead to different limits, then $\lim\limits_{(x,y)\to(0,0)} \dfrac{y^2-4x^2}{2x^2+y^2}$ must not exist



the two easiest paths to check: along $x=a$

along $y=b$

along $x=a=0$

make $x=0$ and turn into a one-variable limit of $y\to b=0$

along $x=0$: $\lim\limits_{(x,y)\to(0,0)} \dfrac{y^2-4x^2}{2x^2+y^2} = \lim\limits_{y\to 0} \dfrac{y^2-4(0)^2}{2(0)^2+y^2} = \lim\limits_{y\to 0} \dfrac{y^2}{y^2} = 1$

along $y=b=0$: $\lim\limits_{(x,y)\to(0,0)} \dfrac{y^2-4x^2}{2x^2+y^2} = \lim\limits_{x\to 0} \dfrac{(0)^2-4x^2}{2x^2+(0)^2} = \lim\limits_{x\to 0} \dfrac{-4x^2}{2x^2} = -2$

Since these two paths lead to different limits,

$$\lim_{(x,y)\to(0,0)} \frac{y^2-4x^2}{2x^2+y^2} \quad \boxed{\text{does not exist}}$$

So, what happens if those two paths led to the same limit? Test other paths?

→ sometimes, or find other ways that are path-independent to find limit

<u>example</u> $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y}$

clearly $\dfrac{x^2-y^2}{x+y}$ does not exist at $(0,0)$, so continuity can't help.

check the two easiest paths: along $x = a$ (0)

along $y = b$ (0)

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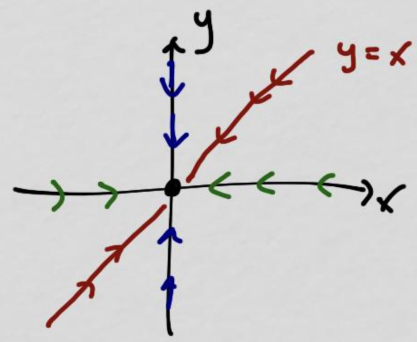
$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y}$$



along $x=0$: $\displaystyle \lim_{y\to 0} \frac{0^2-y^2}{0+y} = \lim_{y\to 0} \frac{-y^2}{y} = 0$

along $y=0$: $\displaystyle \lim_{x\to 0} \frac{x^2-0^2}{x+0} = \lim_{x\to 0} \frac{x^2}{x} = 0$

but this is NOT enough, we need see that ALL paths lead to 0

along $y=x$: $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y} = \lim_{x\to 0} \frac{x^2-x^2}{x+x} = \lim_{x\to 0} \frac{0}{2x} = 0$

Still NOT enough!

what's next? $y=x^2$! and what if that also goes to 0?

find a path-independent way

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$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y} = \lim_{(x,y)\to(0,0)} \frac{\cancel{(x+y)}(x-y)}{\cancel{x+y}} = \lim_{(x,y)\to(0,0)} x-y$$

it is easy to see that no matter how $(x,y)\to(0,0)$,

$x-y$ eventually goes to nearly zero because both x and y eventually become nearly zero

so, therefore, we conclude that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y} = \boxed{0}$

example

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+2y^2)}{x^2+2y^2}$$

does $\frac{\sin(x^2+2y^2)}{x^2+2y^2}$ exist at $(0,0)$? NO.

instead of worrying about paths, let's think of a path-independent way to find limit

as $(x,y)\to(0,0)$, $(x^2+2y^2)\to 0$

so, this limit is the same as $\lim_{u\to 0} \frac{\sin(u)}{u}$ $\quad u = x^2+2y^2$

and we know $\lim_{u\to 0} \frac{\sin(u)}{u} = 1$, so limit is $\boxed{1}$

again, this is done w/o assuming any particular path.