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15.3 Partial Derivatives

recall that if $y = f(x)$, then the derivative of y with respect to x

is $\dfrac{dy}{dx} = f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

this gives us the rate of change of $y = f(x)$ as x changes

if $z = f(x, y)$, then z is affected by x and y. We can isolate the effect of x or y alone on z

→ if we hold one variable constant while allowing the other one to change, what is the rate of change of z with respect to the variable that is changing?

we call this a partial derivative

the partial derivative of $z = f(x,y)$ with respect to x is

$$\frac{\partial f}{\partial x} = f_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

y is NOT changing
so is treated as a constant

the variable in the denominator
or subscript is the "live" variable
(all others are treated as constants)

funny-looking d
for partial derivative

Similarly, the partial derivative of $z = f(x,y)$ with respect to y is

$$\frac{\partial f}{\partial y} = f_y = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

x is NOT changing
so is treated as a constant

f_x and f_y tell us the effect of varying just one variable at a time

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just like with one-variable case, we almost never use the limit definition in practice, we just use the various rules that we know.

Same with higher number of variables, we just need to keep track of which variable is treated as a constant.

<u>example</u> $f(x,y) = x^2 + y^3 + xy$

<u>partial derivative with respect to x</u> is

$$\frac{\partial f}{\partial x} = f_x = \frac{\partial}{\partial x}\left(x^2 + y^3 + xy\right)$$

pretend y is a constant

x is variable

y is contant

"constant"

derivative of $(x)(\text{constant})$

just like $\frac{d}{dx}(2x) = 2$

$$= 2x + 0 + y = \boxed{2x + y}$$

because y^3 is constant

the partial derivative of $f(x,y) = x^2+y^3+xy$ with respect to y is

$$\frac{\partial f}{\partial y} = f_y = \frac{\partial}{\partial y}(x^2+y^3+xy) = 0+3y^2+x = \boxed{3y^2+x}$$

x is constant

<u>example</u> $f(x,y) = x^3\tan(xy)$

y is constant $\quad \frac{\partial f}{\partial x} = f_x = \frac{\partial}{\partial x}\left(x^3 \cdot \tan(xy)\right)$ need to use product rule

$$= (x^3)\frac{\partial}{\partial x}(\tan(xy)) + (\tan(xy))\frac{\partial}{\partial x}(x^3)$$

$$= (x^3)\sec^2(xy)\cdot\underbrace{\frac{\partial}{\partial x}(xy)}_{\text{from Chain Rule}} + (\tan(xy))(3x^2)$$

$$= (x^3)\sec^2(xy)(y) + (\tan(xy))(3x^2)$$

$$= \boxed{x^3y\sec^2(xy) + 3x^2\tan(xy)}$$

$$f(x,y) = x^3\tan(xy)$$

$$\frac{\partial f}{\partial y} = f_y = \frac{\partial}{\partial y}\left(x^3 \tan(xy)\right)$$

x is constant

"constant", so product rule is not needed

$$= x^3\sec^2(xy)\frac{\partial}{\partial y}(xy) = x^3\sec^2(xy)\cdot x$$

$$= \boxed{x^4\sec^2(xy)}$$

<u>example</u> $f(x,y) = e^x \sin y$

y is const. $\frac{\partial f}{\partial x} = f_x = \frac{\partial}{\partial x}\left(e^x \sin y\right) = e^x \sin y$

x is const. $\frac{\partial f}{\partial y} = f_y = \frac{\partial}{\partial y}\left(e^x \sin y\right) = e^x \cos y$

just like one-variable case, we can take derivatives of derivatives

$$\frac{\partial f}{\partial x} = f_x = e^x \sin y \qquad \frac{\partial f}{\partial y} = f_y = e^x \cos y$$

let's take the partial derivative of $\frac{\partial f}{\partial x} = f_x$ with respect to x

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \underbrace{\frac{\partial^2 f}{\partial x^2}}_{\text{take partial with respect to } x \text{ and then again}} = f_{xx} = \frac{\partial}{\partial x}\left(e^x \sin y\right) = e^x \sin y$$

y is constant

similarly, the partial derivative of $\frac{\partial f}{\partial y} = f_y$ with respect to y

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} = \frac{\partial}{\partial y}\left(e^x \cos y\right) = -e^x \sin y$$

x is constant

the partial derivative of $\frac{\partial f}{\partial x} = f_x$ with respect to y

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\underbrace{\partial y \, \partial x}} = f_{xy}$$

in this notation order is LEFT to RIGHT

note that in this notation
the order of variable being "live"
is from RIGHT to LEFT
(first we did partial with respect to x
then with respect to y)

$$= \frac{\partial}{\partial y}\left(e^x \sin y\right) = e^x \cos y$$

similarly, $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\underbrace{\partial x \, \partial y}} = f_{yx} = \frac{\partial}{\partial x}\left(e^x \cos y\right) = e^x \cos y$

note the order
RIGHT → LEFT

note $f_{xy} = f_{yx}$ and this is NOT a coincidence

it turns out that these "mixed partials" are equal wherever $f(x,y)$ is defined and f_x and f_y are continuous (Clairaut's Theorem)

<u>example</u> $f(x,y) = e^{x^2 y}$

y is const. $f_x = \frac{\partial}{\partial x}\left(e^{x^2y}\right) = e^{x^2y}\frac{\partial}{\partial x}(x^2y) = 2xye^{x^2y}$

x is const. $f_y = \frac{\partial}{\partial y}\left(e^{x^2y}\right) = e^{x^2y}\frac{\partial}{\partial y}(x^2y) = x^2e^{x^2y}$

x is const. $\underline{\underline{f_{xy}}} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(2xye^{x^2y}\right)$ product rule needed

$= 2xy\frac{\partial}{\partial y}\left(e^{x^2y}\right) + e^{x^2y}\frac{\partial}{\partial y}(2xy)$

$= 2xy \cdot x^2e^{x^2y} + e^{x^2y} \cdot 2x = 2xe^{x^2y}(x^2y+1)$

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$$f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(x^2 e^{x^2y}\right)$$ product rule needed again

$$= x^2\frac{\partial}{\partial x}\left(e^{x^2y}\right) + e^{x^2y}\frac{\partial}{\partial x}(x^2)$$

$$= x^2 \cdot 2xy\,e^{x^2y} + e^{x^2y}\cdot 2x$$

$$= 2x\,e^{x^2y}\left(x^2y+1\right)$$

so we see $f_{xy} = f_{yx}$ again

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(2xy\,e^{x^2y}\right) = 2xy\cdot e^{x^2y}\cdot 2xy + e^{x^2y}\cdot 2y$$

$$= 2y\,e^{x^2y}\left(2x^2y+1\right)$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y) = \dots$$ (same idea)

functions of more than two variables are handled the same way

example $f(x,y,z) = xyz$

$f_x = yz$

$f_y = xz$

$f_z = xy$

$f_{xy} = z$

$f_{xyz} = 1$

$f_{yx} = z$

$f_{yxz} = 1$

$f_{zy} = x$

$f_{zyx} = 1$

note the mixed partials are again the same

$$f_{xyz} = f_{yxz} = f_{zyx}$$