15.4 The Chain Rule

recall that if $y = f(x)$ and $x = g(t)$ then since y ultimately depends on t, so we must be able to find $\dfrac{dy}{dt}$.

Chain rule: $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt}$ (or $y' = f'(g(t))\,g'(t)$)

how do the variables depend on one another?

dependency tree:

$$\begin{tikzcd}
y \arrow[d, dash] \arrow[d, bend left=60, "\frac{dy}{dx}"] \arrow[dd, bend right=70, "\frac{dy}{dt}"'] \\
x \arrow[d, dash] \arrow[d, bend left=60, "\frac{dx}{dt}"] \\
t
\end{tikzcd}$$

each step down the tree is a derivative

multiply together to get the rate of change of the top with respect to the bottom with functions more variables, the basic idea stays the same

the tree is more important and will have branches.

(because variables can depend on more than one other variable)

each step down is still a derivative, but will be a partial derivative

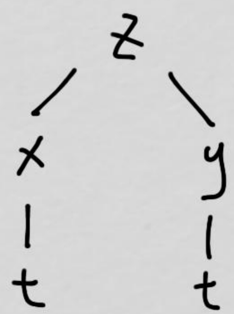
if there is a split (or a branch)

example $z = f(x,y) = x + y^2$

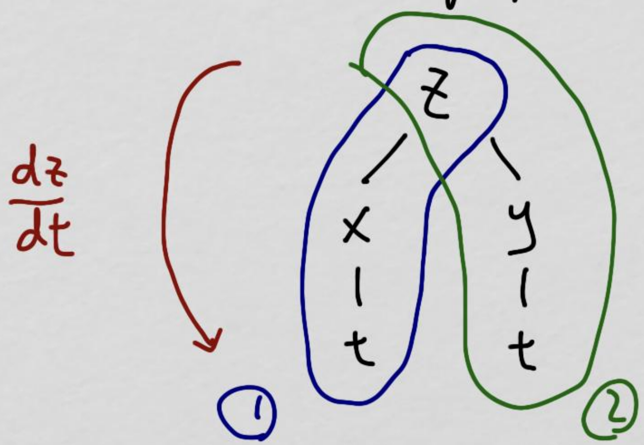
$x = e^t, \quad y = \ln t$

find $\frac{dz}{dt}$ (note it's a regular d because z ultimately depends on just t)

dependency tree:



So, we see as t changes, both branches will contribute to $\frac{dz}{dt}$



two paths to go from z to t: ①, ②

if there is a split, use partial derivative

$$\frac{dz}{dt} = \underbrace{\frac{\partial z}{\partial x}\frac{dx}{dt}}_{①} + \underbrace{\frac{\partial z}{\partial y}\frac{dy}{dt}}_{②}$$

partial because of the split from z to x, y

regular d because no split

$z = x + y^2$ $\qquad z_x = 1, \quad z_y = 2y$

$x = e^t$ $\qquad \frac{dx}{dt} = e^t$

$y = \ln t$ $\qquad \frac{dy}{dt} = \frac{1}{t}$

$$\frac{dz}{dt} = (1)(e^t) + (2y)\left(\frac{1}{t}\right) = e^t + \frac{2y}{t} = \boxed{e^t + \frac{2\ln t}{t}}$$

normally express in terms of the "bottom variable" verify that another way:

$z = x + y^2, \quad x = e^t, \quad y = \ln t$

we can sub out x, y right away

$$z(t) = e^t + (\ln t)^2$$

$$\frac{dz}{dt} = e^t + 2(\ln t)\frac{1}{t} = e^t + \frac{2\ln t}{t}$$ same result

(this method may not practical if there are lots of variables or if the functions are complicated).

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$z = \sin(x+y)$

$x = u^2+v \qquad y = 1-2uv$

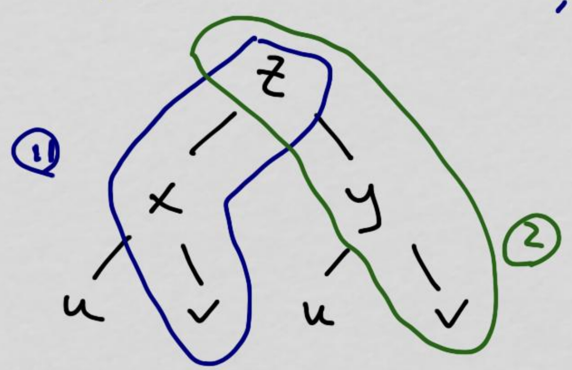
$$= \cos(x+y)(2u) + \cos(x+y)(-2v)$$

$$= 2\cos(x+y)(u-v)$$

express in terms of "bottom variables": u, v

$$= 2\cos(u^2+v+1-2uv)(u-v)$$

$$= \boxed{2(u-v)\cos(u^2+v+1-2uv)}$$



for $\frac{\partial z}{\partial v}$, see the tree on this page

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \cos(x+y)(1) + \cos(x+y)(-2u)$$

$$= \cos(x+y)(1-2u) = \boxed{(1-2u)\cos(u^2+v+1-2uv)}$$

<u>example</u>

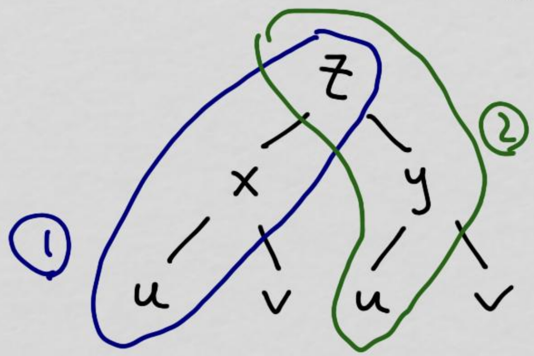
$z = \sin(x+y)$

$x = u^2 + v$

$y = 1 - 2uv$

find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$ (notice "∂" and not "d" because z ultimately depends on two variables: u, v)

tree:



to find $\dfrac{\partial z}{\partial u}$, there are two paths, step down like in last example

split $\rightarrow \partial$ (partials)

$$\frac{\partial z}{\partial u} = \underbrace{\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}}_{1} + \underbrace{\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}}_{2}$$

the Chain Rule can also be used to perform implicit differentiation

example $x^2 + 2y^2 = 4$

if y is an implicit function of x, find $\frac{dy}{dx}$

old way: $\frac{d}{dx}(x^2 + 2y^2) = \frac{d}{dx}(4)$

$$2x + 4y\frac{dy}{dx} = 0$$

so, $\frac{dy}{dx} = -\frac{2x}{4y} = -\frac{x}{2y}$

but we can do this using the Chain Rule

for functions of more than one variable

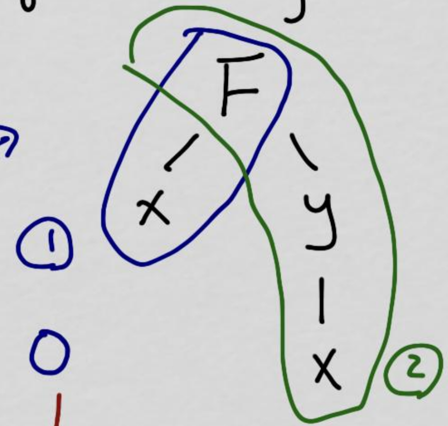
$x^2+2y^2=4$

define $F(x,y)=x^2+2y^2-4=0$

note $F(x,y)$ is really a function of just x, but we don't know what it exactly looks like, since we don't know exactly how y depends on x.

so, let's call the version of $F(x,y)$ that only contains x $f(x)$

$$f(x)=\underbrace{F(x,y)}=0$$



two ways to get from F to x

therefore, $$\frac{df}{dx}=\underbrace{\frac{\partial F}{\partial x}}_{1}+\underbrace{\frac{\partial F}{\partial y}\frac{dy}{dx}}_{2}=0$$

because $f(x)=F(x,y)=0$

so, $\frac{df}{dx}=0=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx}$

then, we can solve $\frac{dy}{dx}$:

Draw Erase Select Point Add Undo Redo Rec Stop View Close

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

(note $f(x)$ is NOT relevant at all)

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$$

back to $x^2+2y^2=4$, find $\frac{dy}{dx}$

define $F = x^2+2y^2-4$

then $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x}{4y} = \boxed{-\frac{x}{2y}}$ same result

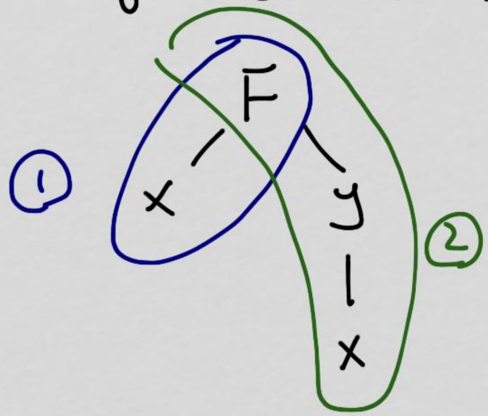
example

$$e^y \sin x = x + xy$$

if y is an implicit function of x, find $\frac{dy}{dx}$

let $F(x,y) = e^y \sin x - x - xy = 0 = \underbrace{f(x)}$

what $F(x,y)$ would look like if y is gone



then $\frac{df}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$

then, again, $\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y} = \boxed{-\frac{e^y \cos x - 1 - y}{e^y \sin x - x}}$

we can do this with functions of more variables, too

example $\quad xy + yz + xz = 3$

if z is an implicit function of x and y

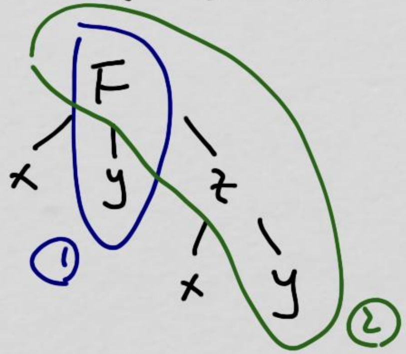
find $\dfrac{\partial z}{\partial y}$

define $F(x,y,z) = xy + yz + xz - 3 = 0 = \underbrace{f(x,y)}$

what $F(x,y,z)$ would look like if we know z as function of x, y

$f(x,y) = F(x,y,z) = 0$

so, two ways to go from F to y to find $\dfrac{\partial F}{\partial y}$



$$\frac{\partial f}{\partial y} = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\boxed{\frac{\partial z}{\partial y}} = 0 \rightarrow \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}} = -\frac{F_y}{F_z}$$

we want this

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \boxed{-\frac{x+z}{y+x}}$$

$$F = xy + yz + xz - 3$$