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15.5 Directional Derivative and the Gradient

f_x and f_y are the rates of change of $f(x,y)$ with respect to x and y, which also tell us how the function is changing if we move in a direction parallel to the x-axis and the y-axis respectively.

if we want to know the rate of change of $f(x,y)$ in a direction not parallel to the axes, but in, for example, the direction $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$?

to find this, we need the Directional Derivative

let $\vec{u} = \langle a, b \rangle$ be a unit vector specifying the direction, then the rate of change of $f(x,y)$ in that direction is given by the Directional Derivative

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$$\boxed{D_{\vec{u}} f(x,y) = \lim_{h\to 0} \frac{f(x+h\underline{a},\ y+h\underline{b}) - f(x,y)}{h}}$$

notice if $\vec{u} = \langle 1, 0\rangle = \vec{i}$ (so, $a=1$, $b=0$)

then $D_{\vec{u}} f(x,y) = \lim_{h\to 0} \dfrac{f(x+h, y) - f(x,y)}{h} = \dfrac{\partial f}{\partial x} = f_x$

this is the partial derivative of f with respect to x

if $\vec{u} = \langle 0, 1\rangle = \vec{j}$ (so, $a=0$, $b=1$)

then $D_{\vec{u}} f(x,y) = \lim_{h\to 0} \dfrac{f(x, y+h) - f(x,y)}{h} = \dfrac{\partial f}{\partial y} = f_y$

so, the directional derivative can be thought of as a generalization of the partial derivatives the limit definition is, of course, not practical.

In practice, we use the following form of the Directional Derivative

$$\boxed{\begin{aligned} D_{\vec{u}} f(x,y) &= \frac{\partial f}{\partial x}(x,y)\,a + \frac{\partial f}{\partial y}(x,y)\,b \\ &= f_x(x,y)\,a + f_y(x,y)\,b \end{aligned}}$$

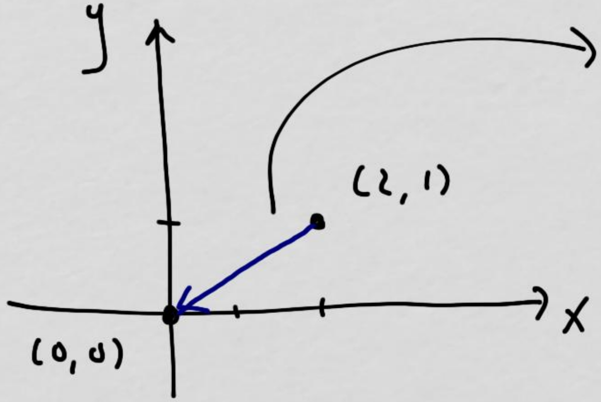
where $\vec{u} = \langle a, b \rangle$ is the <u>unit vector</u> specifying the direction

<u>example</u> $f(x,y) = \cos(2x+3y)$

find the directional derivative at $(2, 1)$

in the direction toward the origin

first, find $\vec{u}$:



this vector is $\langle -2, -1 \rangle$

But we need this to be a unit vector

$$\vec{u} = \frac{\langle -2, -1 \rangle}{|\langle -2, -1 \rangle|}$$

so, $\vec{u} = \left\langle \frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right\rangle$

a b

$f(x,y) = \cos(2x+3y)$

$f_x = -2\sin(2x+3y)$

$f_y = -3\sin(2x+3y)$

$$D_{\vec{u}} f(2,1) = f_x(2,1)\,a + f_y(2,1)\,b$$

$$= -2\sin(7)\cdot\frac{-2}{\sqrt{5}} - 3\sin(7)\cdot\frac{-1}{\sqrt{5}}$$

$$= \boxed{\frac{7}{\sqrt{5}}\sin(7)}$$

this tells us the function $f(x,y) = \cos(2x+3y)$ is increasing at the rate of $\frac{7}{\sqrt{5}}\sin(7)$ in the direction toward the origin from $(2,1)$

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let's look at the directional derivative formula again

$$D_{\vec{u}} f(x,y) = f_x(x,y)\, a + f_y(x,y)\, b$$

we can rewrite is as

$$D_{\vec{u}} f(x,y) = \underbrace{\langle f_x(x,y), f_y(x,y) \rangle}_{\text{this vector is called the Gradient}} \cdot \underbrace{\langle a, b \rangle}_{\text{unit vector } \vec{u} \text{ giving the direction}}$$

the gradient of $f(x,y)$, written as $\vec{\nabla} f(x,y)$, $\nabla f(x,y)$, or $\text{grad } f(x,y)$

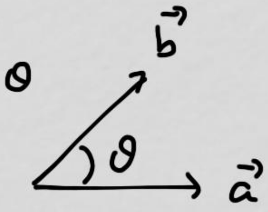
"del" or nabla

$$\boxed{\begin{aligned} \text{gradient: } \vec{\nabla} f(x,y) = \nabla f(x,y) = \text{grad } f(x,y) &= \langle f_x(x,y), f_y(x,y) \rangle \\ &= \frac{\partial f}{\partial x}(x,y)\, \vec{i} + \frac{\partial f}{\partial y}(x,y)\, \vec{j} \end{aligned}}$$

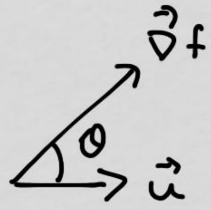
this also tells us that the directional derivative is the **dot product** of the gradient and the unit vector $\vec{u}$

$$\boxed{D_{\vec{u}} f(x,y) = \vec{\nabla} f(x,y) \cdot \vec{u}}$$

recall $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$



so, $D_{\vec{u}} f(x,y) = |\vec{\nabla} f(x,y)|\, \underbrace{|\vec{u}|}_{\substack{1 \text{ because} \\ \vec{u} \text{ is unit vector}}} \cos\theta$



$= |\vec{\nabla} f(x,y)| \cos\theta$ → between -1 and 1

$D_{\vec{u}} f(x,y)$ is **maximized** when $\vec{\nabla} f$ and $\vec{u}$ are in the same direction $(\theta = 0)$

because $\cos 0 = 1$

$D_{\vec{u}} f(x,y)$ is **minimized** when $\vec{\nabla} f$ and $\vec{u}$ are in the opposite directions $(\theta = \pi)$

because $\cos\pi = -1$

if $\vec{\nabla} f$ and $\vec{u}$ are orthogonal, then $\theta = \frac{\pi}{2}$ and $\cos\frac{\pi}{2} = 0$, so

$$D_{\vec{u}} f(x,y) = 0$$

another way to say all of this:

the direction of the gradient vector is the direction of maximum directional derivative, the magnitude of the gradient vector is the maximum directional derivative

example $f(x,y) = 5 - x^2 - y^2$

starting at $(0,2)$, in what direction is $f(x,y)$ increasing most rapidly? And in what direction is $f(x,y)$ staying unchanged?

$f(x,y) = 5 - x^2 - y^2$

gradient : $\vec{\nabla} f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = \langle -2x, -2y \rangle$

$$\vec{\nabla} f(0,2) = \langle 0, -4 \rangle$$

this is the direction to go to see the maximum directional derivative

so, we want to go in the direction $\langle 0, -4 \rangle$ (or $\langle 0, -1 \rangle$ as a unit vector) to see the most rapid increase in $f(x,y)$

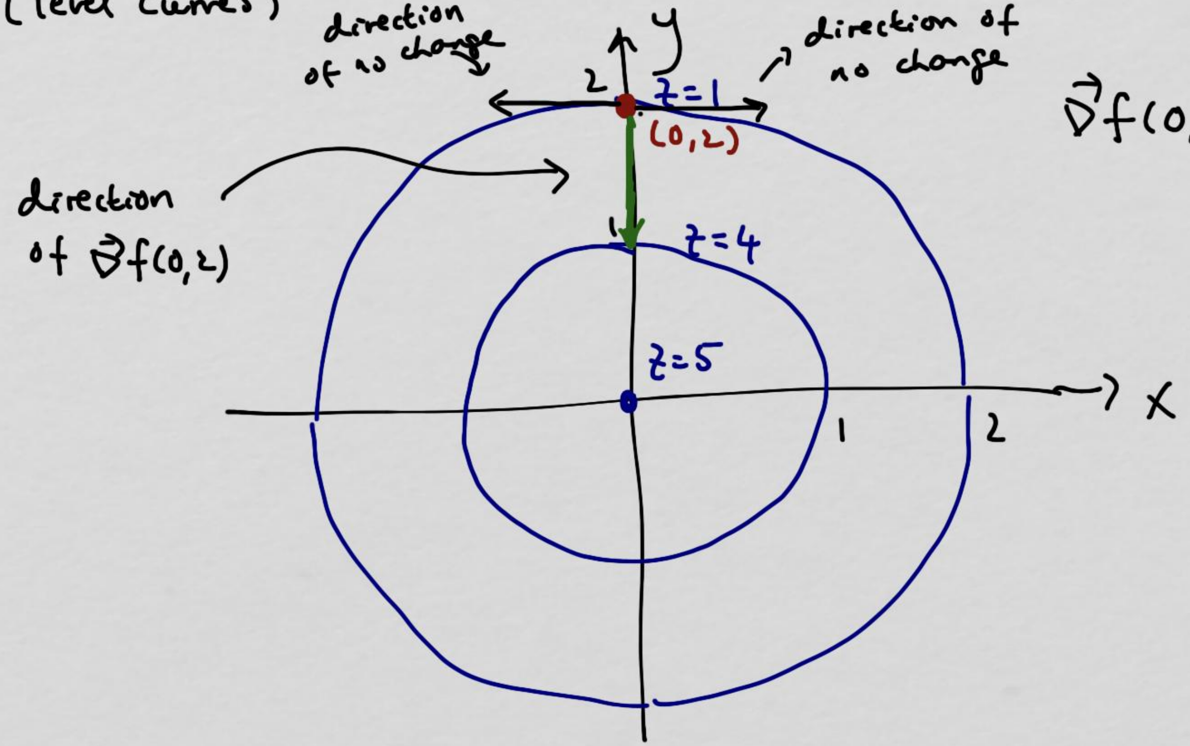
we will see no change in the value of $f(x,y)$ if we go in a direction orthogonal to the gradient

$\rightarrow$ so, in $\langle 1, 0 \rangle$ or $\langle -1, 0 \rangle$

(since $\langle 1, 0 \rangle \cdot \langle 0, -4 \rangle = 0$ and $\langle -1, 0 \rangle \cdot \langle 0, -4 \rangle = 0$)

let's look at this graphically (level curves)

$$f(x,y) = 5 - x^2 - y^2$$



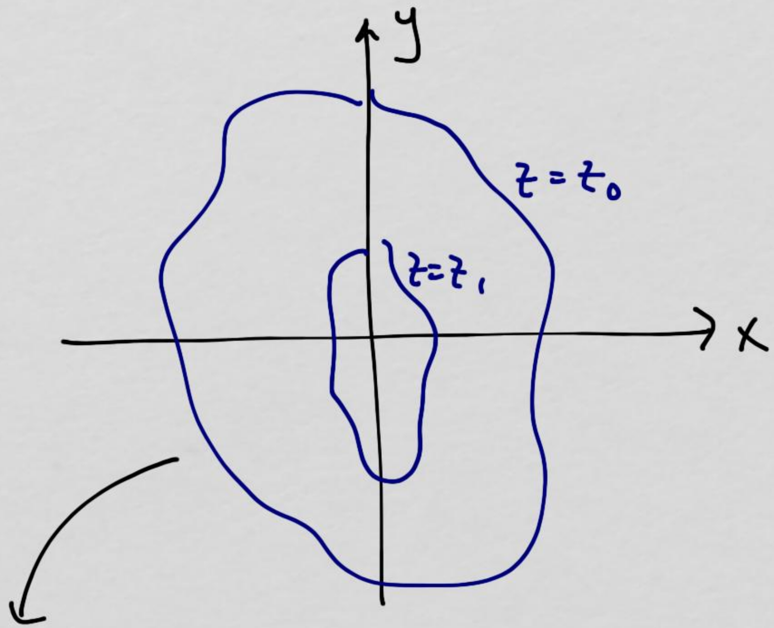
$$\vec{\nabla} f(0,2) = \langle 0, -4 \rangle$$

it looks like to get the maximum directional derivative, we want to go orthogonal to the a level curve and to get no change, we want to go tangent to a level curve

can we verify that?

let's see if that observation is correct

suppose $z = z_0 = f(x,y)$ are level curves of $z = f(x,y)$



we can consider this level curve as a space curve described by

$\vec{r}(t) = \langle x(t), y(t) \rangle$

so, $z = z_0 = f(x,y) = f(x(t), y(t))$

then $z_0 = f(x(t), y(t))$ can be rewritten as

$F(x,y) = f(x(t), y(t)) - z_0 = 0$ (the Chain Rule set up we used last time)

$$\begin{array}{ccc} & F & \\ \swarrow & & \searrow \\ x & & y \\ | & & | \\ t & & t \end{array}$$

so, $F(x,y)$ can be written as a function of t, call it $g(t)$

then $$\frac{dg}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} = 0$$ because $F = g(t) = 0$

or $$\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = 0$$

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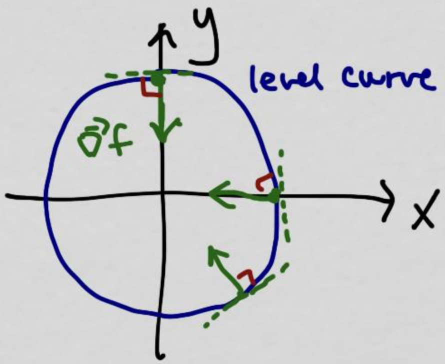
and since $F(x,y) = f(x,y) - z_0$, $\dfrac{\partial F}{\partial x} = \dfrac{\partial f}{\partial x}$, $\dfrac{\partial F}{\partial y} = \dfrac{\partial f}{\partial y}$

so, the last expression is also

$$\underbrace{\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle}_{\vec{\nabla} f} \cdot \underbrace{\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle}_{\text{is the tangent to } \vec{r}(t) = \langle x(t), y(t) \rangle} = 0$$

so, this says the gradient is orthogonal to the tangent vector on a level curve



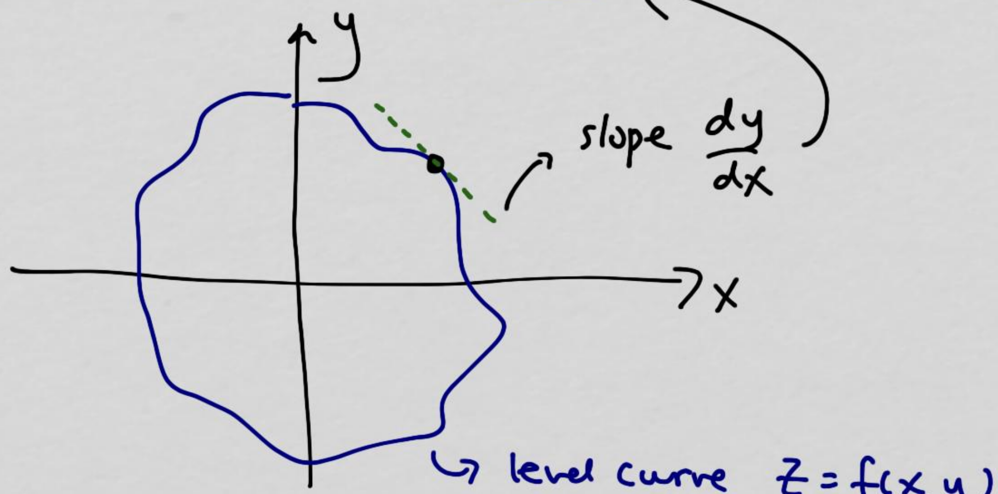
furthermore, from $\langle f_x, f_y \rangle \cdot \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle = 0$

we get $f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = 0$

or $$\frac{dy}{dt} = - \frac{f_x \frac{dx}{dt}}{f_y}$$

or $$\frac{dy/dt}{dx/dt} = - \frac{f_x}{f_y} \longrightarrow \boxed{\frac{dy}{dx} = - \frac{f_x}{f_y}}$$

this tells us the slope of the curve curve



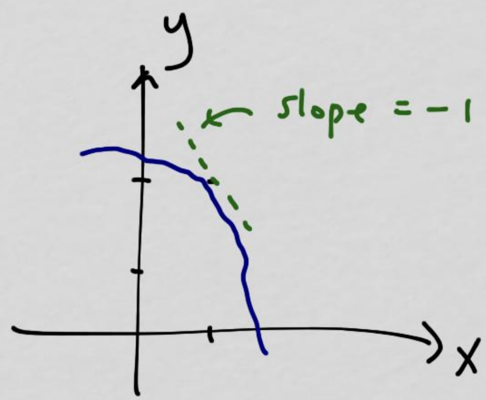
<u>example</u> $f(x,y) = x e^{y}$

find the slope of the level curve at $(1,2)$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{e^{y}}{x e^{y}} = -\frac{1}{x}$$

evaluate at $(1,2)$ we get

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,2)} = -\frac{1}{1} = -1$$



if $z = f(x,y)$, then each $z = z_0$ results in a level curve

and $\vec{\nabla} f$ is orthogonal to the level curve

if $w = f(x,y,z)$ (a function of 3 variable), each

$w = w_0$ results in a level surface and $\vec{\nabla} f$ is still

orthogonal to the level surface

as we go higher in dimensions, each "slice" of the shape

also goes higher in dimensions, but $\vec{\nabla} f$ is always

orthogonal to it.

example

$f(x,y,z) = xyz$

find the equation of the plane tangent to the level surface at $(1,2,3)$

$f(x,y,z) = xyz$ is in $\mathbb{R}^4$ (3 independent variables x, y, z and 1 dependent variable $f(x,y,z)$)

it's a 4D object

each "slice" at $f = K$ is a 3D object (surface)

$f(1,2,3) = 6$ $\quad$ $6 = xyz$ is a surface (level surface)

$\vec{\nabla} f = \langle f_x, f_y, f_z \rangle = \langle yz, xz, xy \rangle$

$$\underbrace{\vec{\nabla} f(1,2,3) = \langle 6, 3, 2 \rangle}_{\text{orthogonal to the level surface containing point } (1,2,3)}$$

which gives us all we need for a tangent plane : normal vector
point

so, the tangent plane as equation

$$\vec{\nabla} f(1,2,3) \cdot \langle x-1,\ y-2,\ z-3 \rangle = 0$$

$$\langle 6,\ 3,\ 2 \rangle \cdot \langle x-1,\ y-2,\ z-3 \rangle = 0$$

$$\boxed{6x+3y+2z-18=0}$$