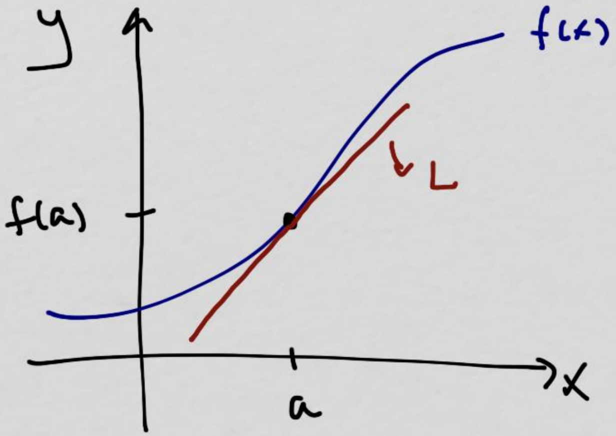
15.6 Tangent Plane and Linear Approximation

recall if $y = f(x)$, then the tangent line to $f(x)$ at $(a, f(a))$ can be written as $L = f(a) + f'(a)(x-a)$



near $x = a$, the tangent line is approximately equal to the true curve $f(x)$

if x is close to a, then

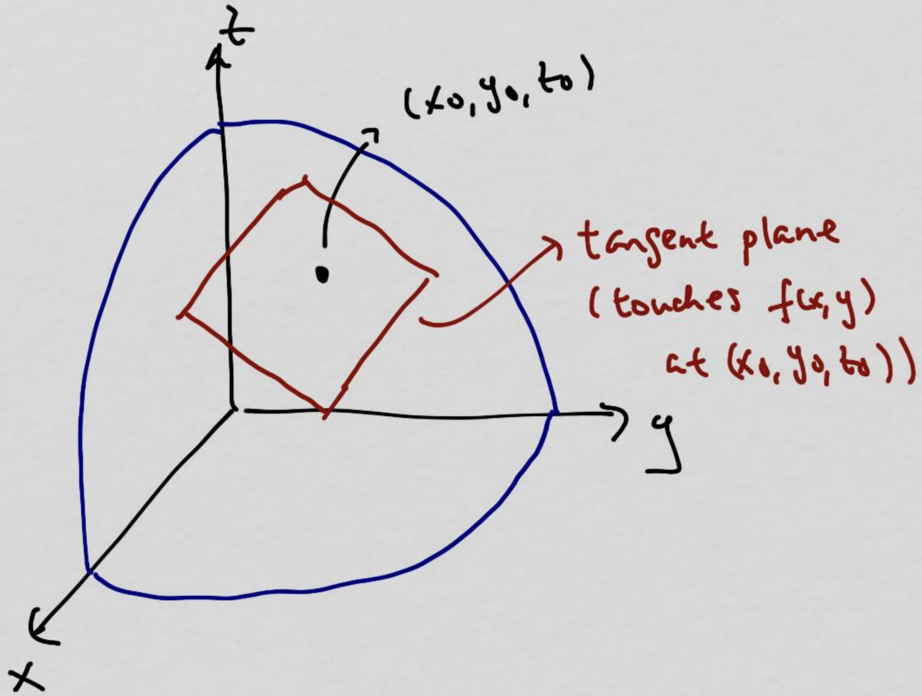
$$L = f(a) + f'(a)(x-a) \approx f(x)$$

Linear approximation of $f(x)$ near $x = a$

now we'll try to do the same for $z = f(x,y)$

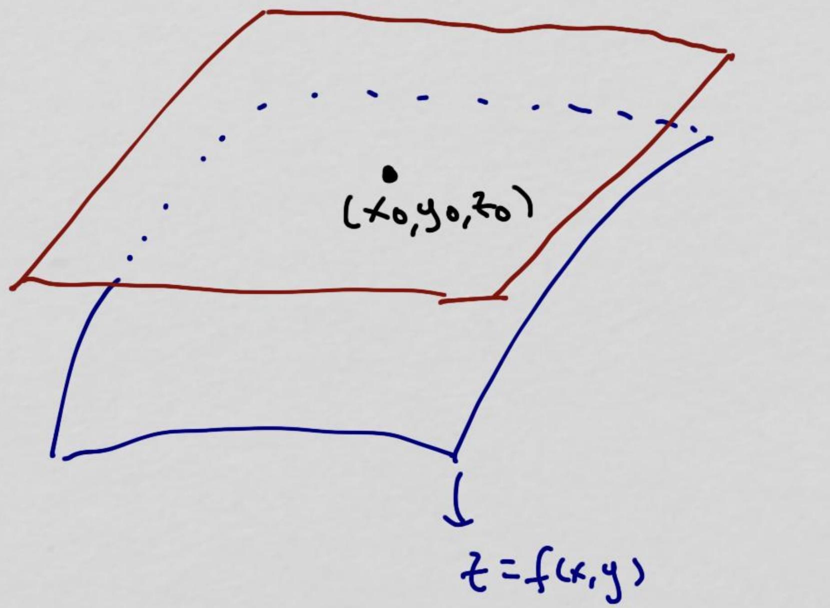
$z = f(x,y)$ is a surface, so instead of a tangent line at (x_0, y_0, z_0), we actually have infinitely many tangent lines, and they form a

<u>tangent plane</u>

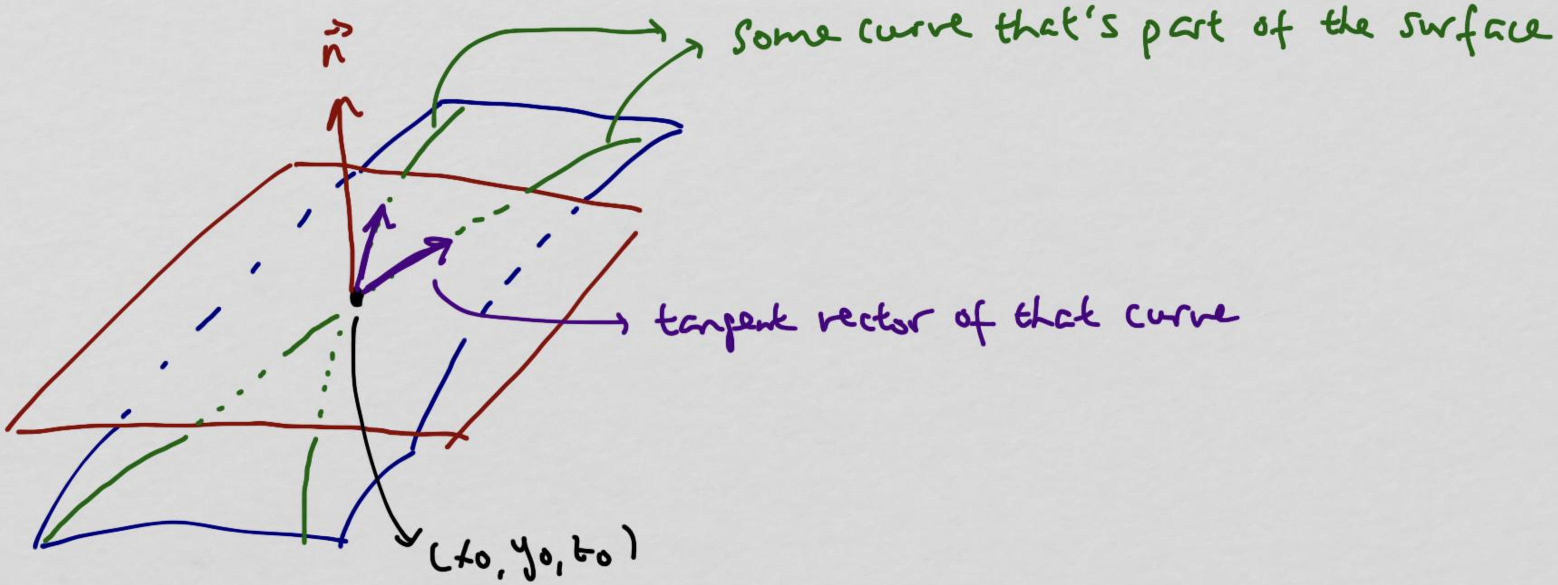


very near the point (x_0, y_0, z_0)

the tangent plane $\approx$ true surface



How do we find the equation of the tangent plane?



there are infinitely-many curves that form the surface, call them

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

and their tangent vectors $\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$ form the tangent plane

and the normal vector of the tangent plane is orthogonal to all of them

$\rightarrow$ find the normal vector, then w/ the point $(x_0, y_0, z_0) \rightarrow$ plane equation if $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a function of t, then

the equation of the surface is also $z(t) = f(x(t), y(t))$

let $F(x,y,z) = f(x(t), y(t)) - z(t) = 0 = \underbrace{G(t)}$

what F looks like if only t remains

$$\begin{tikzcd}[column sep=small] & F \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ x \arrow[d, dash] & y \arrow[d, dash] & z \arrow[d, dash] \\ t & t & t \end{tikzcd}$$

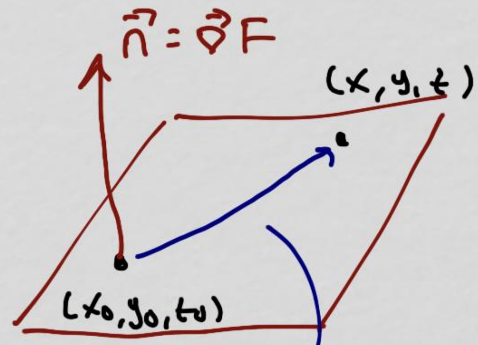
because $F = 0 = G(t)$

so, $\dfrac{dG}{dt} = \underbrace{\dfrac{\partial F}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial F}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial F}{\partial z}\dfrac{dz}{dt}} = 0$

$$\underbrace{\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle}_{\vec{\nabla} F} \cdot \underbrace{\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle}_{\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle} = 0$$

tangent vector of all curves that form $z = f(x,y)$

$\vec{\nabla} F \cdot \vec{r}\,'(t) = 0$ means the normal vector of the tangent plane is $\vec{\nabla} F$



$\vec{\nabla} F = \langle F_x, F_y, F_z \rangle$

point on the tangent plane : (x_0, y_0, z_0)

so, equation of the tangent plane is

$$\boxed{\begin{aligned} &F_x (x - x_0) + F_y (y - y_0) + F_z (z - z_0) = 0 \\ &\text{or} \quad \langle F_x, F_y, F_z \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0 \end{aligned}}$$

where $F(x,y,z) = f(x,y) - z$

if we prefer to use the explicit equation of the surface $z = f(x,y)$ then we just need to modify the above equation a little bit

$F = f - z \rightarrow F_x = f_x, \quad F_y = f_y, \quad F_z = -1$

$$\boxed{\begin{aligned} &f_x (x - x_0) + f_y (y - y_0) - (z - z_0) = 0 \\ &\text{or} \quad z - z_0 = f_x (x - x_0) + f_y (y - y_0) \end{aligned}}$$

example $\quad z = f(x,y) = \sqrt{x^2+y^2}$

find the tangent plane at $(3,4,5)$

let $\quad F = f - z = \sqrt{x^2+y^2} - z$

then $\vec{\nabla} F = \langle F_x, F_y, F_z \rangle$ is normal to the tangent plane

$$= \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1 \right\rangle$$

$$\vec{\nabla} F(3,4,5) = \left\langle \frac{3}{5}, \frac{4}{5}, -1 \right\rangle$$

so, the tangent plane equation is

$$\left\langle \frac{3}{5}, \frac{4}{5}, -1 \right\rangle \cdot \langle x-3, y-4, z-5 \rangle = 0$$

$$\boxed{\frac{3}{5}(x-3) + \frac{4}{5}(y-4) - (z-5) = 0}$$

close to (x_0, y_0, z_0) the tangent plane $\approx f(x,y)$

let's see how good the approximation is with the previous example

$$z = f(x,y) = \sqrt{x^2+y^2} \text{ near } (3, 4, 5)$$

tangent plane: $z - 5 = \frac{3}{5}(x-3) + \frac{4}{5}(y-4)$

so, close to $(3,4,5)$, $f(x,y) \approx z = 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4)$

how close is the approximation?

let's try $x = 3.01$, $y = 3.99$

approximation from tangent plane: $z \approx 5 + \frac{3}{5}(3.01-3) + \frac{4}{5}(3.99-4)$
$= 4.998$

true value: $f(3.01, 3.99) = \sqrt{(3.01)^2 + (3.99)^2} = 4.99802$ not bad

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the other form: $z - z_0 = f_x (x - x_0) + f_y (y - y_0)$

notice it resembles the tangent line equation for $y = f(x)$

$y - f(a) = f'(a)(x - a)$

$f(x,y) = \sqrt{x^2 + y^2}$

$f_x = \dfrac{x}{\sqrt{x^2+y^2}}$ $\qquad f_y = \dfrac{y}{\sqrt{x^2+y^2}}$

at $x = 3,\ y = 4,\ z = 5$

$f_x = \dfrac{3}{5}, \quad f_y = \dfrac{4}{5}$

so, equation of the tangent plane is

$$\boxed{z - 5 = \frac{3}{5}(x - 3) + \frac{4}{5}(y - 4)}$$

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back to $z - z_0 = f_x(x - x_0) + f_y(y - y_0)$

if $x - x_0$ is very small, then $x - x_0 = dx$ (small change in x)

and $y - y_0 = dy$, $z - z_0 = dz$

so, the tangent plane equation turns into

$$dz = f_x\, dx + f_y\, dy$$

and this can be used to estimate the change in z given changes in x and y.

(the smaller dx and dy are, the better the approximation of dz is) example

$$z = f(x,y) = x^2 y$$

If x starts at 1 and increases by 0.01
and y starts at 3 and decreases by 0.09

by how much z changes?

$$dz = f_x\,dx + f_y\,dy$$

$$= (2xy)\,dx + (x^2)\,dy$$

0.01 -0.09 (because y decreases)

these x and y
refer to the original/stating x, y
(because that's where the tangent plane is built)

$$= (2\cdot 1\cdot 3)(0.01) + (1^2)(-0.09) = \boxed{-0.03}$$

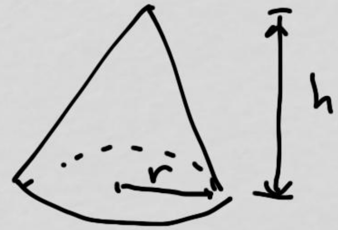
we expect z to decrease by about 0.03

how good is the approximation?

true change in z: $[z \text{ at new } (x,y)] - [z \text{ at old } (x,y)]$

$$= \underbrace{(1.01)^2(2.91)}_{\text{new } z} - \underbrace{(1)^2(3)}_{\text{old } z} = -0.031509$$

<u>example</u> The volume of a cone is $V = \frac{1}{3}\pi r^2 h$



if r is increased by 1% and h is decreased by 3%, estimate the percent change in volume (V).

$$V = \frac{1}{3}\pi r^2 h$$

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh = \left(\frac{2}{3}\pi r h\right)dr + \left(\frac{1}{3}\pi r^2\right)dh$$

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notice we don't know dr and dh because we don't know the current r, h

the change of r is 1% but that is NOT dr

$$dV = \left(\frac{2}{3}\pi r h\right) dr + \left(\frac{1}{3}\pi r^2\right) dh$$

we don't want this, we want is the % change in V

$$\text{\% change of something} = \frac{\text{change in that something}}{\text{that something}}$$

so, % change of V is $\frac{dV}{V}$ (if V had changed from 10 to 9, then the % change in V is $\frac{9-10}{10} = -\frac{1}{10}$

$= -0.1$

$= -10\%$)

divide $dV = \left(\frac{2}{3}\pi r h\right) dr + \left(\frac{1}{3}\pi r^2\right) dh$ by $V = \frac{1}{3}\pi r^2 h$

$$\frac{dV}{V} = \frac{\frac{2}{3}\pi r h}{\underbrace{\frac{1}{3}\pi r^2 h}_{V}} dr + \frac{\frac{1}{3}\pi r^2}{\underbrace{\frac{1}{3}\pi r^2 h}_{V}} dh$$

$$= \frac{2}{r} dr + \frac{1}{h} dh$$

$$= 2\underbrace{\left(\frac{dr}{r}\right)}_{\substack{\text{\% change in } r \\ \text{which is } 1\%}} + \underbrace{\left(\frac{dh}{h}\right)}_{\substack{\text{\% change in } h \\ \text{which is } -3\%}}$$

$$= 2(0.01) + (-0.03) = -0.01 = \text{decrease by } 1\%$$

so, the resulting % change in V is a decrease of 1%