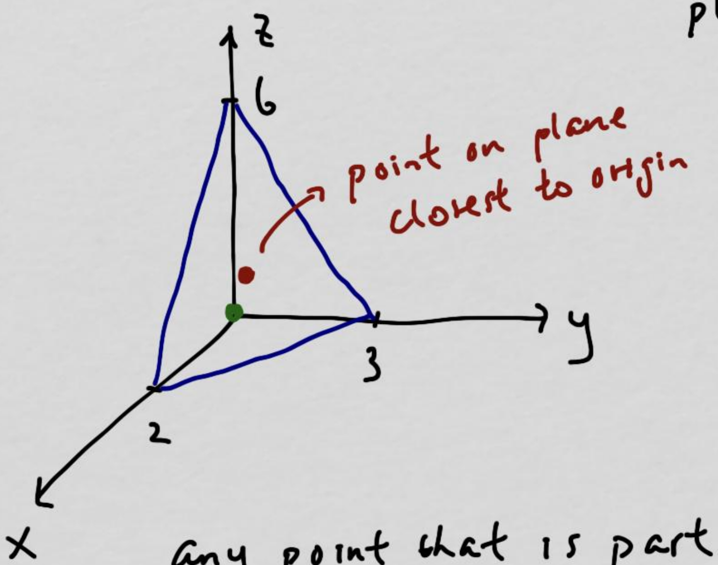
15.7 Maximum and Minimum Problems (continued)

application of the 2nd Derivative Test

example Which point on the plane $3x+2y+z=6$ is the closest to the origin?



plane x-int: $3x=6 \to x=2$

y-int: $2y=6 \to y=3$

z-int: $z=6$

any point that is part of the plane has coordinates

$$(x,\ y,\ \underbrace{6-3x-2y}_{\text{from } 3x+2y+z=6})$$

$$f_x = 2x + 2(6-3x-2y)(-3) = 0 \longrightarrow 5x + 3y = 9 \quad \text{—} \; (1)$$

$$f_y = 2y + 2(6-3x-2y)(-2) = 0 \longrightarrow 6x + 5y = 12 \quad \text{--} \; (2)$$

solving (1) and (2) simultaneously, we get

$$x = \frac{9}{7}, \quad y = \frac{6}{7} \qquad \boxed{\text{critical point : } \left(\frac{9}{7}, \frac{6}{7}\right)}$$

$$\left.\begin{aligned} f_{xx} &= 20 \\ f_{yy} &= 10 \\ f_{xy} &= 12 \end{aligned}\right\} \quad D = f_{xx}f_{yy} - (f_{xy})^2 = 200 - 144 = 56$$

at $\left(\frac{9}{7}, \frac{6}{7}\right)$, $D > 0$, $f_{xx} > 0$, so $f = d^2$ (and therefore d) is at a local minimum

$$z = 6 - 3x - 2y = \frac{3}{7}$$

so, the point closest to the origin is $\boxed{\left(\frac{9}{7}, \frac{6}{7}, \frac{3}{7}\right)}$

distance between that point and the origin is

$$d = \sqrt{(x-0)^2 + (y-0)^2 + (6-3x-2y-0)^2}$$

$$d = \sqrt{x^2 + y^2 + (6-3x-2y)^2}$$

to use the 2nd Deriv. Test, we need d_x, d_y, d_{xx}, d_{yy}, d_{xy}

but that radical will make things messy

since $d \geq 0$, the values of (x,y) that minimizes d will be the same as the values of (x,y) that minimizes the square of d

so we can minimize d^2 instead

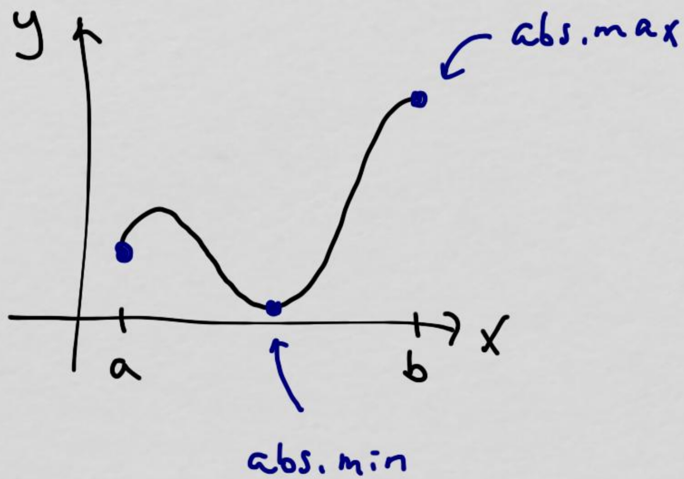
let $f = d^2 = x^2 + y^2 + (6-3x-2y)^2$

now f_x, f_y, etc are much easier to find

The 2nd Derivative Test gives us relative or local max/min.

Let's now find out how to find the absolute max/min.

recall if $y = f(x)$, $a \le x \le b$, then the absolute max/min of $f(x)$ can occur at the critical points or at the boundary



example : $y = x^2 \quad -1 \le x \le 3$

$y' = 2x = 0 \rightarrow x = 0$

now compare y at

$\underset{\text{critical pt}}{x=0}$, $\underbrace{x=-1,\ x=3}_{\text{end points}}$

at $x=0$, $y=0$ abs. min

at $x=-1$, $y=1$

at $x=3$, $y=9$ abs. max

note the boundary is now a collection of curves and not just points

in general, if the projection of $z = f(x,y)$ is a region D,

then we need to find the critical points within D, and locations

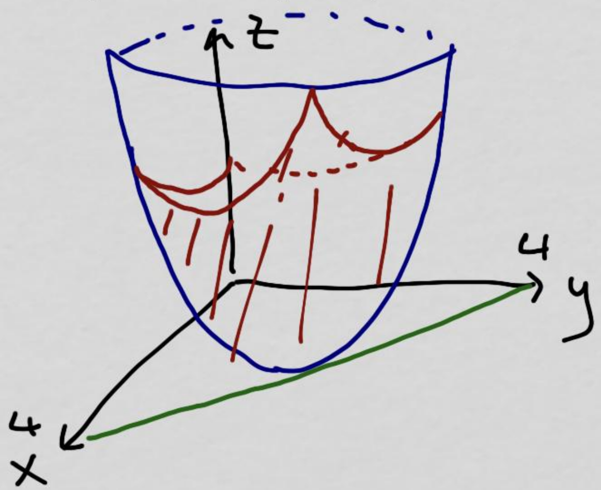
of max/min on the boundary of D, then compare the values of $z = f(x,y)$

at those places.

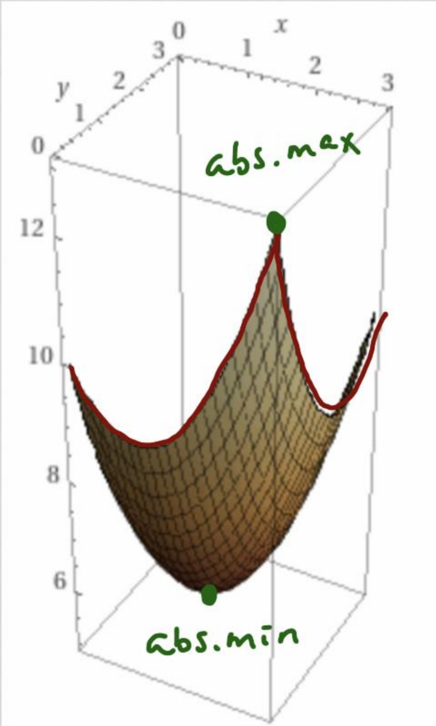
<u>example</u> $z = f(x,y) = x^2 + y^2 - 2x - 4y + 10$

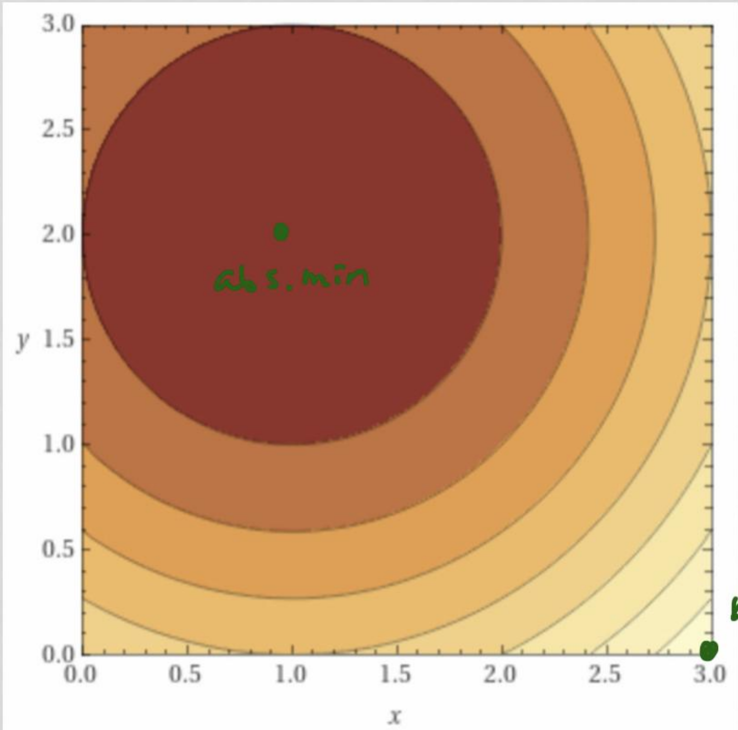
above the region that is a triangle with vertices at

$(0,0)$, $(0,4)$, $(4,0)$



find the highest/lowest points on the surface that is part of the paraboloid $x^2+y^2-2x-4y+10$ whose "shadow" on the xy-plane is a triangle when $z = f(x,y)$ we do something very similar





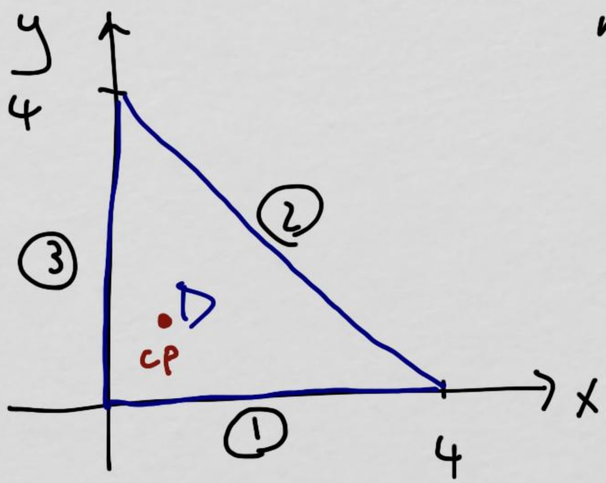
abs. max

contour: lighter color represents higher $f(x,y)$

$0 \le x \le 3$

$0 \le y \le 3$

the abs. max/min will still occur at the interior critical points or at the boundary let's look at the region D (the "shadow")



we can break up the boundary of D into

① $y = 0, \quad 0 \le x \le 4$

② $y = 4 - x, \quad 0 \le x \le 4$

③ $x = 0, \quad 0 \le y \le 4$

find critical points of $f(x,y) = x^2 + y^2 - 2x - 4y + 10$ within D

$f_x = 2x - 2 = 0 \rightarrow x = 1$

$f_y = 2y - 4 = 0 \rightarrow y = 2$

$\boxed{\text{Critical point } (1, 2)}$ inside D? yes

if not, discard from consideration

next, find out where on each edge of D, does $f(x,y)$ attain max/min on that edge.

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$$f(x,y) = x^2 + y^2 - 2x - 4y + 10$$

① $y = 0, \quad 0 \le x \le 4$

becomes a one-variable abs. max/min problem

$f(x,y)$ becomes $\boxed{f(x) = x^2 - 2x + 10 \quad 0 \le x \le 4}$

$f'(x) = 2x - 2 = 0 \to x = 1$

end points: $x = 0, \ x = 4$

the points where f might have max/min are:

$\boxed{(0,0), (4,0), (1,0)}$

② $y = 4 - x, \quad 0 \le x \le 4$

$f(x,y)$ becomes $f(x) = x^2 + (4-x)^2 - 2x - 4(4-x) + 10$

$\boxed{f(x) = 2x^2 - 6x + 10 \quad 0 \le x \le 4}$

$f' = 4x - 6 = 0 \to x = 3/2$

end points: $x = 0, \ x = 4$

points of interest: $\boxed{(\underline{0,4}), (\underline{4,0}), (\underline{3/2, 5/2})}$

$y = 4 - x$

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③ $x=0, \quad 0 \le y \le 4$

$f(x,y)$ becomes $\boxed{f(y) = y^2 - 4y + 10 \qquad 0 \le y \le 4}$

$f' = 2y - 4 = 0 \rightarrow y = 2$

end points: $y=0, \ y=4$

points of interest: $\boxed{(0,0), \ (0,4), \ (0,2)}$

now we compare values of $f(x,y) = x^2 + y^2 - 2x - 4y + 10$ at the points we collected:

critical pt $f(1,2) = 5$

① $\begin{cases} f(0,0) = 10 \\ f(4,0) = 18 \\ f(1,0) = 9 \end{cases}$

② $\begin{cases} f(0,4) = 10 \\ f(3/2, 5/2) = 11/2 \end{cases}$

(4,0) is duplicate of ①

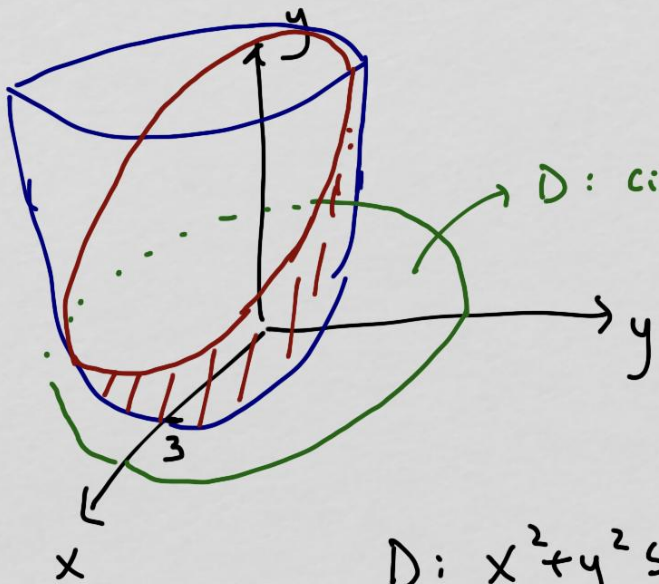
③ ← $f(0,2) = 6$

(0,0), (0,4) duplicate ①, ②

$\boxed{\begin{array}{l} \text{abs. max of } 18 \text{ at } (4,0) \\ \text{abs. min of } 5 \text{ at } (1,2) \end{array}}$

<u>example</u> Find abs. max/min of $f(x,y) = x^2+y^2-6x+9$

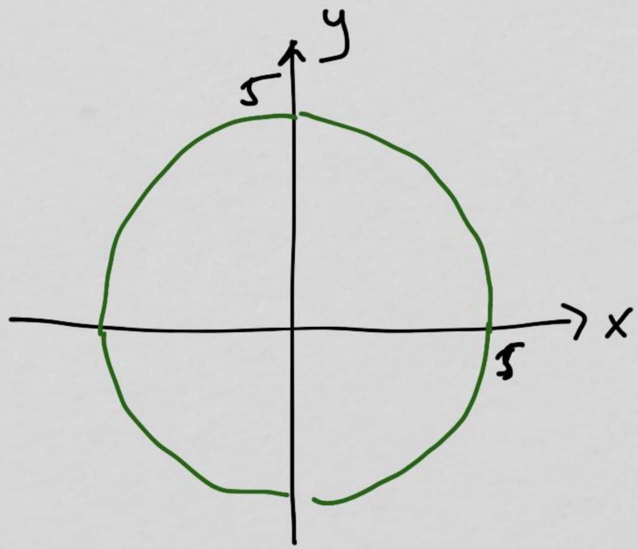
above the region $D: \{(x,y) : x^2+y^2 \leq 25\}$



D: circle of radius 5

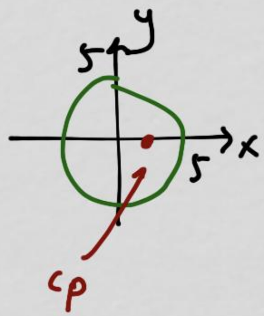
find highest/lowest points on the part of the paraboloid that casts a shadow that is $x^2+y^2 \leq 25$

$D: x^2+y^2 \leq 25$



notice we can't divide it into straight edges.

let's find the interior critical points first



$$f(x,y) = x^2+y^2-6x+9$$

$$\left.\begin{array}{l} f_x = 2x-6 = 0 \rightarrow x = 3 \\ f_y = 2y = 0 \rightarrow y = 0 \end{array}\right\} \boxed{\text{Critical pt : } (3,0)}$$

inside D? yes.

on the boundary of D, $\underbrace{x^2+y^2=25, \quad -5 \le x \le 5}_{\text{sub into } f(x,y)}$ $\left(-\sqrt{25-x^2} \le y \le \sqrt{25-x^2}\right)$

$f(x,y)$ becomes

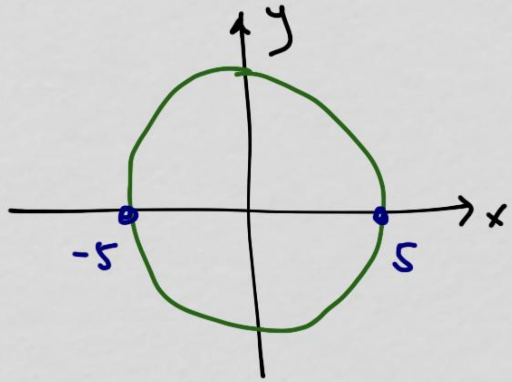
$$f(x,y) = \underbrace{x^2+y^2}_{25}-6x+9 \longrightarrow f(x) = 34-6x \quad -5 \le x \le 5$$

$f' = -6 \neq 0$ no critical point

f on edge can only attain max/min at ends of $-5 \le x \le 5$

So, the points of interest on the edge are

$$\boxed{(-5,0),\ (5,0)}$$



together with the critical pt $(3,0)$

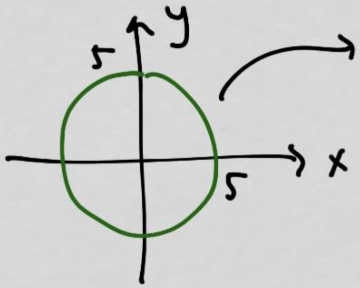
we compare $f(x,y) = x^2+y^2-6x+9$

$f(3,0) = 0 \to$ abs. min at $(3,0)$

$f(-5,0) = 64 \to$ abs. max at $(-5,0)$

$f(5,0) = 4$

an alternative way to find the points of interest on the edge is to express the edge as a space curve



$\longrightarrow x^2+y^2=25 \longrightarrow \vec{r}(t) = \langle 5\cos t,\ 5\sin t \rangle \quad 0 \le t \le 2\pi$

$5\cos t \to x$, $5\sin t \to y$

now sub these into $f(x,y)$

$$f(x,y) = x^2 + y^2 - 6x + 9$$

$$x = 5\cos t \,, \quad y = 5\sin t \qquad 0 \le t \le 2\pi$$

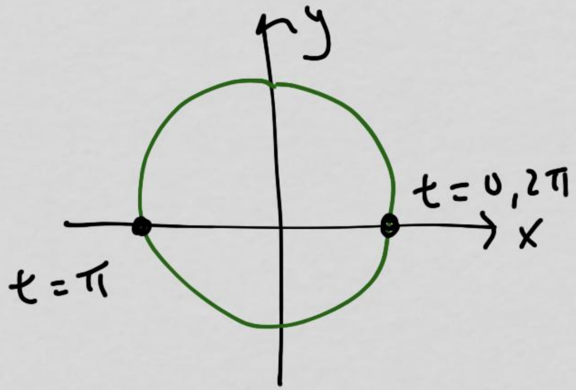
$f(x,y)$ becomes

$$f(t) = 25\cos^2 t + 25\sin^2 t - 30\cos t + 9$$

$$f(t) = 34 - 30\cos t$$

$$f' = 30\sin t = 0 \longrightarrow \quad t = 0, \; t = \pi, \; t = 2\pi$$

which means, the points of interest on the edge are



at $t = 0, 2\pi \rightarrow (5, 0)$

at $t = \pi \rightarrow (-5, 0)$

same as from the other method.