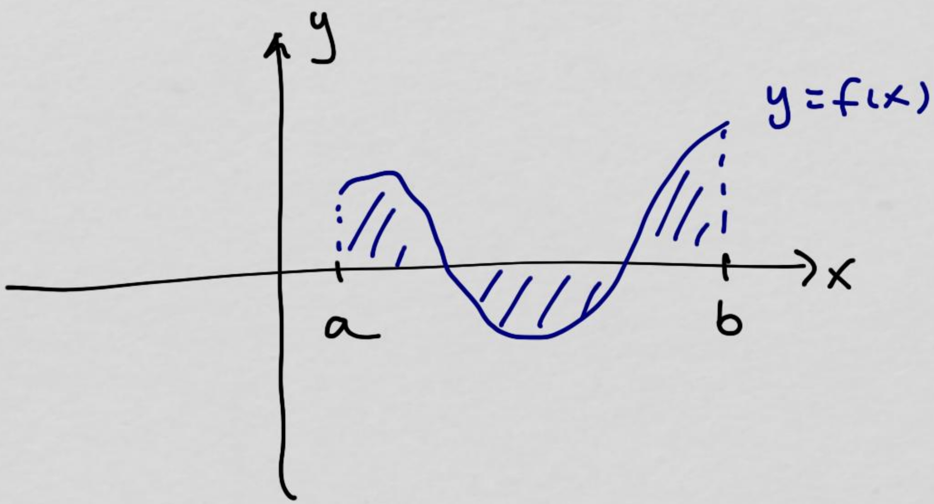
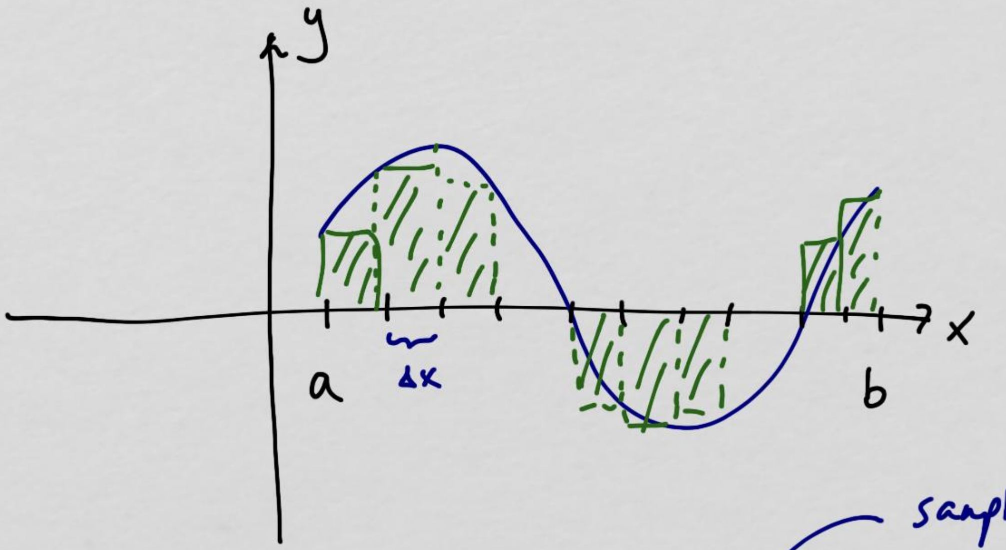
16.1 Double Integrals over Rectangular Regions

if $y = f(x)$, $a \le x \le b$

$\int_a^b f(x)\,dx$ is the net area of the region between $f(x)$ and the x-axis from $x = a$ to $x = b$



which really came from a sum of infinitely-many rectangles (Riemann Sum)



sample point

each rectangle has area $\underbrace{f(x_i)}_{\text{height}}\Delta x$ width of rectange

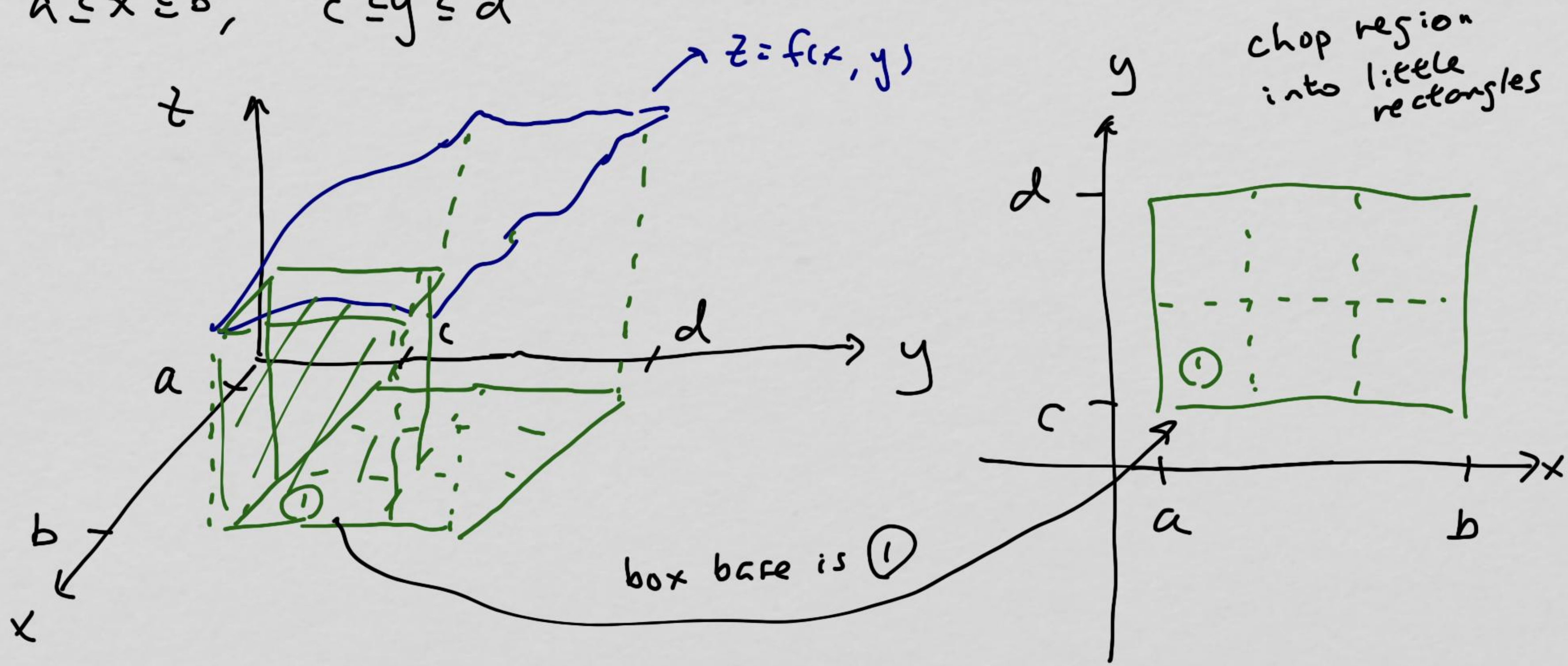
the total is $\sum_{i=1}^{n} f(x_i)\Delta x$ when $n \to \infty$ $\Delta x \to dx$, it becomes

the definite integral: $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

if $z = f(x,y)$ then the idea is fundamentally the same: sum rectangles but now z is a surface, so we are summing rectangular boxes under the surface

in 16.1, we focus on a rectangular region under $z = f(x,y)$

$a \le x \le b$, $\quad c \le y \le d$



if we sum up volumes of all little boxes, we find the approximate volume under $z = f(x,y)$

just like we sum areas of rectangles in $y = f(x)$ case

the sum can represent a volume if $z = f(x,y) \geq 0$ but it doesn't have to.

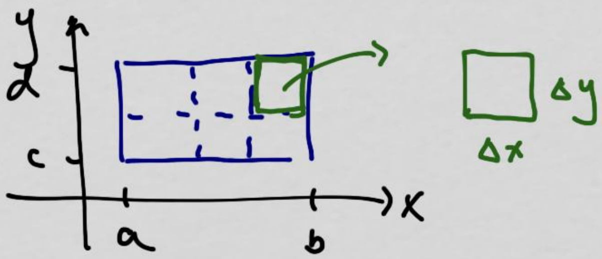
sample points of the grid $R = \{(x,y) : a \leq x \leq b,\ c \leq y \leq d\}$

$$\text{total} \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\,\Delta A$$

← sum over y ($c \leq y \leq d$ divided into m parts)

sum over x ($a \leq x \leq b$ divided into n subintervals)

ΔA is the area of each rectangular base



area $= \Delta A = \Delta x \Delta y = \Delta y \Delta x$

when summing, we vary the variables one at a time

(when sum x, keep y constant)

now we shrink the base area of each box and have more boxes

$n, m \to \infty, \quad \Delta A \to dA$

$$\boxed{\lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_i)\,\Delta A = \int_a^b \int_c^d f(x,y)\, \underbrace{dy\, dx}_{dA}}$$

y bounds

x bounds

if the region R is a rectangle (bounds of x and y are all constants)

then the order of summation / integration is irrelevant so, the integral on the last page is also

$$\int_c^d \int_a^b f(x,y)\,dx\,dx$$

how to evaluate this iterated integral? (double integral)

→ integrate inside-out keeping the other variable constant

<u>example</u>

$$\int_0^1 \underbrace{\int_0^2 (3-x-y)\,dy}\,dx$$

do the inside first

here, y is variable, x is constant

$$= \int_0^1 \left(3y - xy - \frac{1}{2}y^2 \Big|_{y=0}^{y=2}\right) dx$$

$$= \int_0^1 \left[3\cdot 2 - x\cdot 2 - \tfrac{1}{2}(2)^2 - 0 \right] dx$$

$$= \int_0^1 (4-2x)\,dx = 4x - x^2 \Big|_0^1 = \boxed{3}$$

note both bounds are constants for both variables, so the order doesn't change things (equivalent to sum one row or one column at a time)

check: $\displaystyle\int_0^2 \int_0^1 (3-x-y)\,dx\,dy$

again, evaluate inside-out

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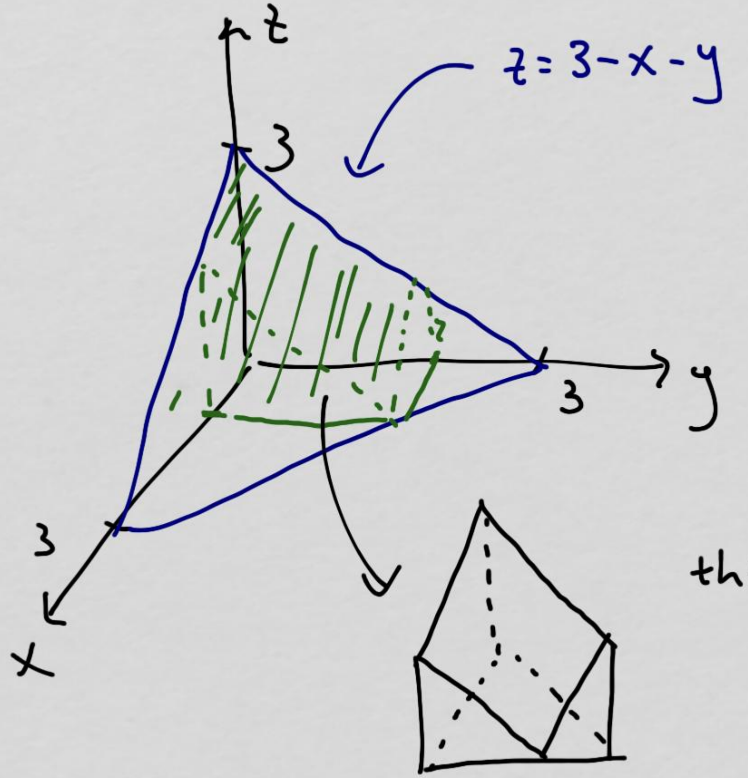
$$\int_0^2 \underbrace{\int_0^1 (3-x-y)\,dx}_{x \text{ is variable, } y \text{ is constant}} dy$$

$$= \int_0^2 \left(3x - \tfrac{1}{2}x^2 - yx \Big|_{x=0}^{x=1}\right) dy = \int_0^2 \left(3 - \tfrac{1}{2} - y\right) dy$$

$$= \int_0^2 \left(\tfrac{5}{2} - y\right) dy = \tfrac{5}{2}y - \tfrac{1}{2}y^2 \Big|_0^2 = 5 - 2 = \boxed{3}$$

this example was calculating the volume under $z = 3 - x - y$

above the rectangle $R = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 2\}$



this trapetoidal wedge
has volume of 3

example

$$\int_2^3 \underbrace{\int_0^{\pi/2} y\cos x \, dx}_{\text{integrate inside-out (so, } x \text{ first)}} \, dy$$

integrate inside-out (so, x first)

y is constant

$$= \int_2^3 \left(y \cdot \sin x \Big|_{x=0}^{x=\pi/2} \right) dy = \int_2^3 \left(y \cdot \sin\frac{\pi}{2} - y \cdot \sin 0 \right) dy$$

$$= \int_2^3 y \, dy = \frac{1}{2} y^2 \Big|_2^3 = \boxed{\frac{5}{2}}$$

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note we could also do this

$$\int_2^3 \int_0^{\pi/2} y \cdot \cos x \, dx \, dy$$

"constant" in first integration (with x)
so it can be pulled out of the inside integral

$$= \int_2^3 y \cdot \left(\int_0^{\pi/2} \cos x \, dx \right) dy = \int_2^3 y \cdot \left(\sin x \Big|_{x=0}^{x=\pi/2} \right) dy$$

$$= \int_2^3 y \cdot \sin \tfrac{\pi}{2} \, dy = \frac{y^2}{2} \Big|_2^3 = \boxed{\frac{5}{2}}$$

example

$$\int_0^1 \underbrace{\int_0^2 y^5 x^2 e^{x^3 y^3}\,dx}\,dy$$

x is variable, y is constant

can be pulled out of inside integral

$$= \int_0^1 y^5 \left[\underbrace{\int_0^2 x^2 e^{x^3y^3}\,dx} \right] dy$$

constant

substitution: $u = x^3 y^3$

$du = 3x^2 y^3\,dx$

$$= \int_0^1 y^5 \left[\int_{x=0}^{x=2} \frac{1}{3y^3} e^u\,du \right] dy = \int_0^1 \frac{1}{3} y^2 \left[\int_{x=0}^{x=2} e^u\,du \right] dy$$

can pull out

$$= \int_0^1 \frac{1}{3} y^2 \left[e^{u} \Big|_{x=0}^{x=2} \right] dy$$

$x^3 y^3$

$$= \int_0^1 \frac{1}{3} y^2 \left[e^{x^3 y^3} \Big|_{x=0}^{x=2} \right] dy = \int_0^1 \frac{1}{3} y^2 \left(e^{8y^3} - 1 \right) dy$$

$$= \underbrace{\int_0^1 \frac{1}{3} y^2 e^{8y^3} \, dy}_{\substack{\text{Sub: } v = 8y^3 \\ dv = 24y^2\,dy}} - \underbrace{\int_0^1 \frac{1}{3} y^2 \, dy}_{\text{easy}}$$

$\vdots$

$\vdots$

$$= \frac{1}{72}\left(e^8 - 1\right) - \frac{1}{9} = \boxed{\frac{1}{72} e^8 - \frac{1}{8}}$$

example $\iint_R x^2\cos(xy)\,dA \qquad R=\{(x,y):\ 0\le x\le \frac{\pi}{3},\ 0\le y\le 1\}$

no order is specified : we choose

let's examine both possible orders

X first (x inside) : $\displaystyle\int_0^1 \underbrace{\int_0^{\pi/3} x^2\cos(xy)\,dx}_{y \text{ is constant}}\,dy$

integrate by parts $u=x^2 \quad dv=\cos(xy)\,dx$

$du=2x\,dx \quad v=\frac{1}{y}\sin(xy)$

and we need TWO ROUNDS of integration by parts

seems hard, try the other order

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y first (y inside) : $\displaystyle \int_0^{\pi/3} \underbrace{\int_0^1 x^2 \cos(xy)\,dy}_{\text{easy, because } x \text{ is constant}}\,dx$

$$= \int_0^{\pi/3} x^2 \left[\int_0^1 \cos(xy)\,dy \right] dx$$

$$= \int_0^{\pi/3} x^2 \left[\frac{1}{x} \sin(xy) \Big|_{y=0}^{y=1} \right] dx$$

$$= \int_0^{\pi/3} x \sin(x)\,dx$$

still requires integration by parts : $u = x$ $\quad dv = \sin(x)\,dx$

$du = dx$ $\quad v = -\cos(x)\,dx$

but only one round

$$= -x\cos(x)\Big|_0^{\pi/3} + \int_0^{\pi/3} \cos(x)\,dx = \boxed{-\frac{\pi}{6} + \frac{\sqrt{3}}{2}}$$

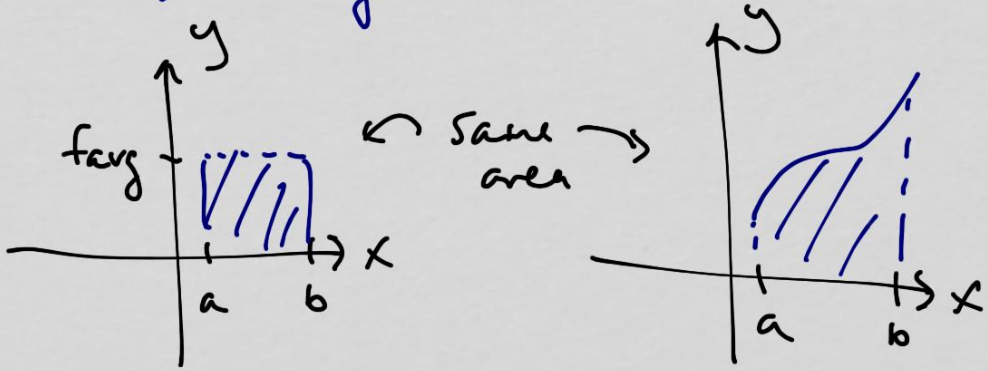
ALWAYS CONSIDER CHANGING INTEGRATION ORDER IF IT LOOKS INTIMIDATING.

recall that $y = f(x) \quad a \le x \le b$

then the average value of $f(x)$, f_{avg}, over the interval $a \le x \le b$

is $\quad f_{avg} = \dfrac{1}{b-a} \displaystyle\int_a^b f(x)\,dx$

$$\Rightarrow \underbrace{(f_{avg})(b-a)}_{\substack{\text{area of rectangle} \\ \text{width } b-a \\ \text{height } f_{avg}}} = \underbrace{\int_a^b f(x)\,dx}_{\substack{\text{area under} \\ f(x)}}$$



if $z = f(x,y)$, that idea carries over, except area → volume, length → area

$z = f(x,y)$ $\quad a \le x \le b, \quad c \le y \le d$

$$\underbrace{(f_{avg})(A)}_{\text{volume of box w/ base area } A \text{ height } f_{avg}} = \underbrace{\iint_R f(x,y)\,dA}_{\text{volume under } z=f(x,y) \text{ above } R}$$

area of region R

$$\boxed{f_{avg} = \frac{1}{A}\iint_R f(x,y)\,dA}$$