16.2 Double Integrals over General Regions

last time: double integrals over rectangular

$$R = \{ (x,y) : a \le x \le b, \quad c \le y \le d \}$$

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

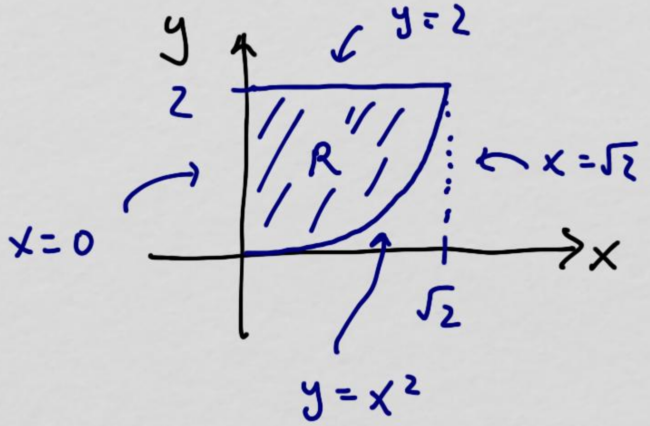
and the order could be switched directly

however, if the region is not rectangle (integration limits are not all constants) then the order is important and cannot be switched arbitrarily.

<u>example</u>

$f(x,y) = xy^2$ $\quad$ $R = \{ (x,y) : 0 \le x \le \sqrt{2},\ x^2 \le y \le 2 \}$

bottom $y = x^2$ $\quad$ top : $y=2$



Basic rule : integrate the variable with constant bounds LAST

(it's on the outside)

here, x is bounded by constants, so it's the outside integral

$$\int_0^{\sqrt{2}} \int_{x^2}^{2} xy^2 \, dy \, dx$$

x $\quad$ y

$$= \int_0^{\sqrt{2}} x \frac{y^3}{3}\Big|_{y=x^2}^{y=2} dx = \int_0^{\sqrt{2}} \left(\frac{8}{3}x - \frac{x^7}{3}\right) dx = \frac{8}{6}x^2 - \frac{1}{24}x^8 \Big|_0^{\sqrt{2}}$$

$$= \frac{8}{3} - \frac{16}{24} = \boxed{2}$$

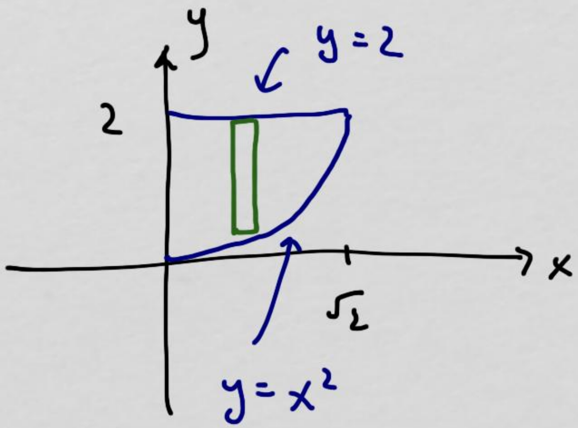
what if we had the order wrong?

$$\int_{x^2}^{2} \int_0^{\sqrt{2}} x y^2 \, dx \, dy = \int_{x^2}^{2} \frac{1}{2} x^2 y^2 \Big|_{x=0}^{x=\sqrt{2}} dy = \int_{x^2}^{2} y^2 \, dy$$

$$= \frac{y^3}{3}\Big|_{x^2}^{2} = \frac{8}{3} - \frac{x^6}{3}$$ ← this x won't go away

we can still switch integration order, but it needs to be done properly by re-formulating the region R

as given: $R = \{ (x,y) : 0 \le x \le \sqrt{2}, \quad x^2 \le y \le 2 \}$

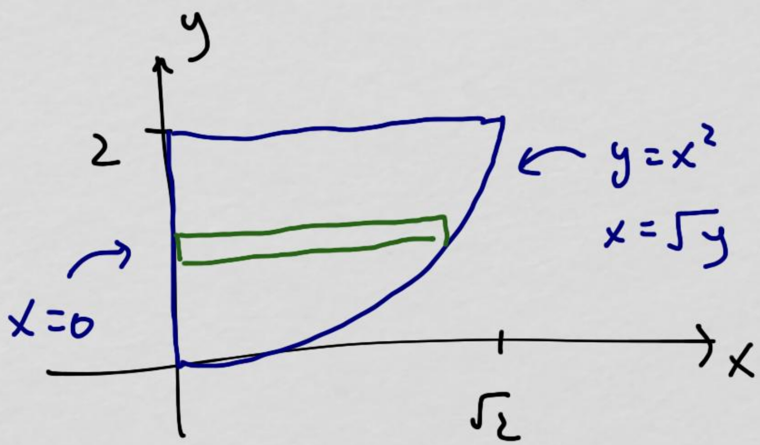


this is a Type I region: x bounded by constants

(if finding area, the typical element is vertical)

(we integrate the height of rectangle as it slides from $x=0$ to $x=\sqrt{2}$)

we can change it to a Type II region: y bounded by constants



rectangle can slide from $y=0$ to $y=2$

$\Rightarrow \quad 0 \le y \le 2$

"height" of rectangle is $\underset{\text{right}}{\sqrt{y}} - \underset{\text{left}}{0}$

$\Rightarrow \quad 0 \le x \le \sqrt{y}$

so, as a Type II: $R = \{ (x,y) : \underbrace{0 \le y \le 2}_{\text{outside integral}}, \underbrace{0 \le x \le \sqrt{y}}_{\text{inside}} \}$

$$\int_0^2 \int_0^{\sqrt{y}} x y^2 \, dx \, dy$$

$y \quad x$

$$= \int_0^2 \tfrac{1}{2} x^2 y^2 \Big|_{x=0}^{x=\sqrt{y}} dy = \int_0^2 \tfrac{1}{2} y^3 \, dy = \tfrac{1}{8} y^4 \Big|_0^2 = \boxed{2}$$

same answer

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<u>example</u>

$$\int_0^{\pi/4} \int_x^{\pi/4} \cos(2x-y)\,dy\,dx$$

as written: $0 \le x \le \pi/4$ Type I

$x \le y \le \pi/4$

$$\int_0^{\pi/4} \left[\int_x^{\pi/4} \cos(2x-y)\,dy \right] dx$$

$u = 2x - y$

$du = -dy$

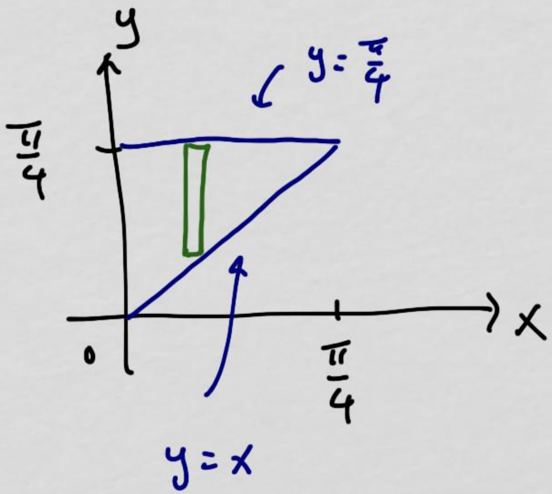
$$= \int_0^{\pi/4} \left[\int_x^{2x-\pi/4} -\cos(u)\,du \right] dx = \int_0^{\pi/4} -\sin(u) \Big|_{u=x}^{u=2x-\pi/4} dx$$

$$= \int_0^{\pi/4} \left[-\sin\left(2x-\frac{\pi}{4}\right) + \sin(x)\right] dx = \underbrace{\int_0^{\pi/4} -\sin\left(2x-\frac{\pi}{4}\right) dx}_{\substack{v = 2x - \frac{\pi}{4} \\ dv = 2\,dx}} + \underbrace{\int_0^{\pi/4} \sin(x)\, dx}_{\text{easy}}$$

$$= \cdots = \frac{1}{2}\cos\left(2x-\frac{\pi}{4}\right) - \cos(x) \Big|_0^{\pi/4} = \cancel{\frac{1}{2}\cos\left(\frac{\pi}{4}\right)} - \cos\left(\frac{\pi}{4}\right) - \cancel{\frac{1}{2}\cos\left(-\frac{\pi}{4}\right)} + \cos(0)$$

$$= \boxed{1 - \frac{1}{\sqrt{2}}}$$

now let's try it with the other order

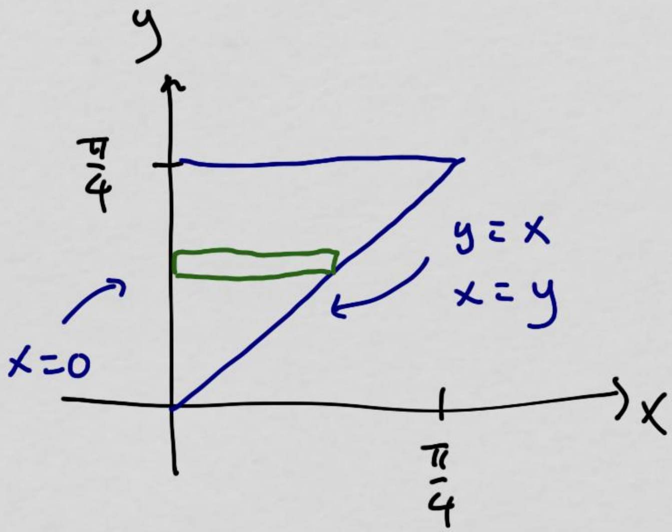


as written:

$$0 \le x \le \frac{\pi}{4}$$

$$x \le y \le \frac{\pi}{4}$$

now change to Type II



rectangle can slide from $y=0$ to $y=\frac{\pi}{4}$

$$0 \le y \le \frac{\pi}{4}$$

"height" = right − left

$= y - 0$

$$\underbrace{0}_{\text{left}} \le x \le \underbrace{y}_{\text{right}}$$

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$$\int_{0}^{\pi/4}\int_{0}^{y} \cos(2x-y)\,dx\,dy = \int_{0}^{\pi/4} \frac{1}{2}\sin(2x-y)\Big|_{x=0}^{x=y}\,dy$$

$y \qquad x$

$$= \int_{0}^{\pi/4}\left[\frac{1}{2}\sin(y) - \frac{1}{2}\sin(-y)\right]dy$$

$\sin(-y) = -\sin(y)$

$$= \int_{0}^{\pi/4} \sin(y)\,dy = -\cos(y)\Big|_{0}^{\pi/4} = \boxed{-\frac{1}{\sqrt{2}} + 1}$$

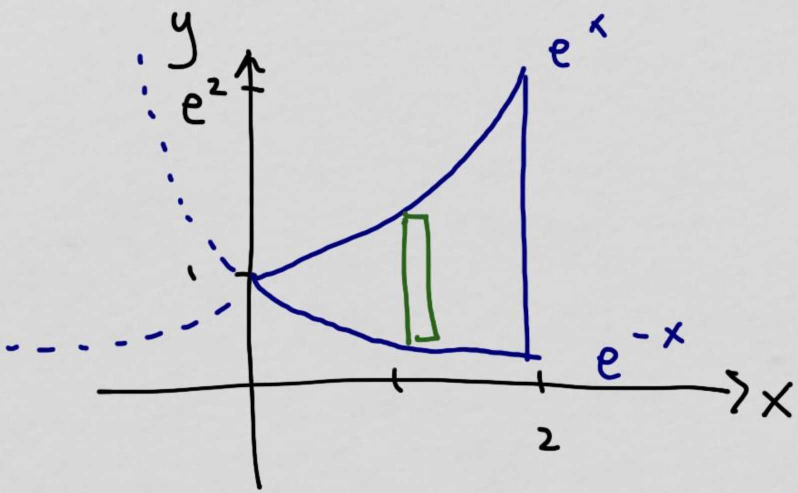
Same answer

<u>example</u> Rewrite $\int_0^2 \int_{e^{-x}}^{e^x} f(x,y)\,dy\,dx$ in the other order

as written: $0 \le x \le 2$

$e^{-x} \le y \le e^x$

Type I (vertical rectangle)



now express as type II

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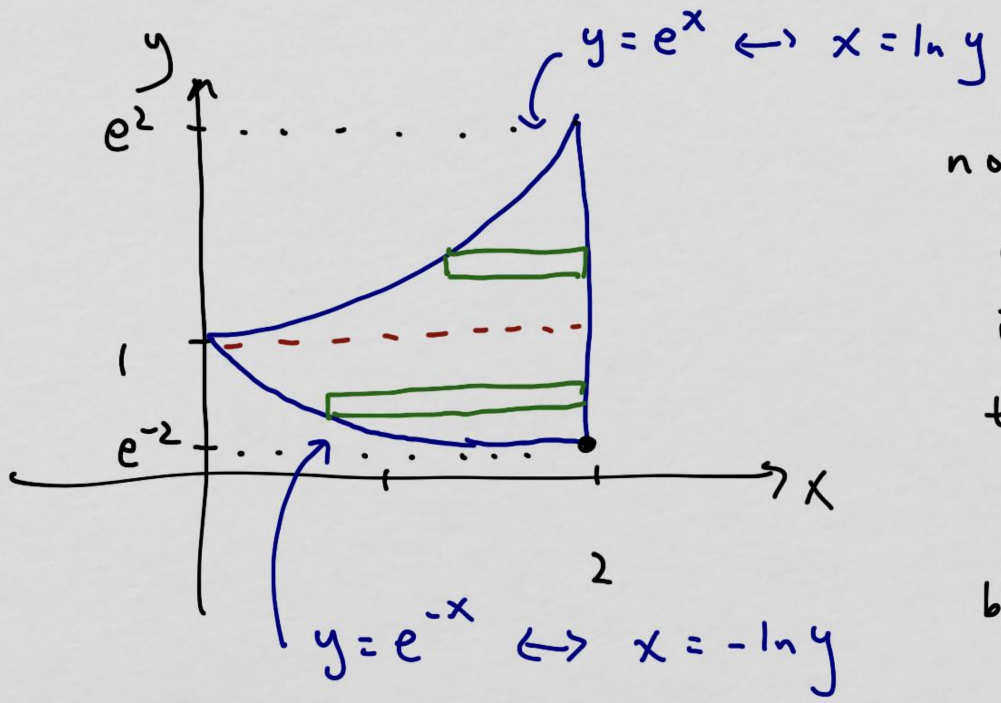


note at $y=1$, the left curve changes, so two integrals are needed

top part: $1 \le y \le e^2$

$\ln y \le x \le 2$

bottom part: $e^{-2} \le y \le 1$

$-\ln y \le x \le 2$

integrals: $$\underbrace{\int_{e^{-2}}^{1} \int_{-\ln y}^{2} f(x,y)\,dx\,dy}_{\text{bottom part}} + \underbrace{\int_{1}^{e^2} \int_{\ln y}^{2} f(x,y)\,dx\,dy}_{\text{top part}}$$

why switch? sometimes we need to if $f(x,y)$ can't be integrated as written

<u>example</u>

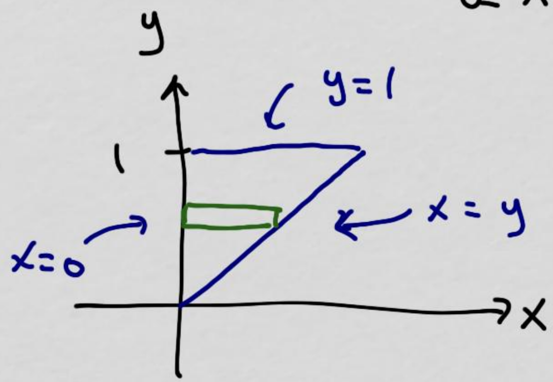
$$\underbrace{\int_0^1 \int_0^y e^{x^2}\,dx\,dy}_{\text{cannot be done}}$$

$$\int e^{x^2}\,dx = ?$$

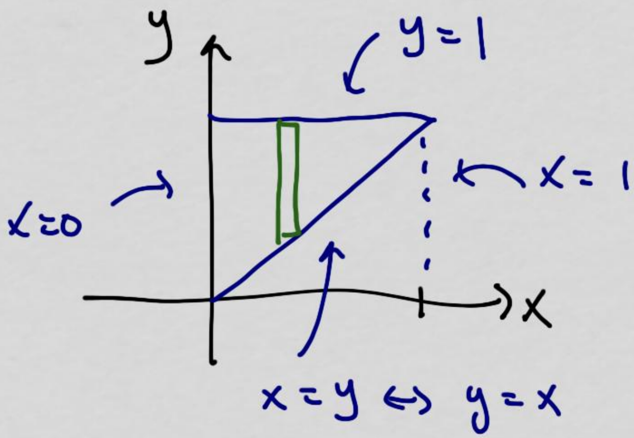
as written $0 \le y \le 1$

$0 \le x \le y$

type <u>II</u>



so, we <u>must</u> switch



now $0 \le x \le 1$

$x \le y \le 1$

$$\int_0^1 \int_x^1 e^{x^2}\,dy\,dx$$

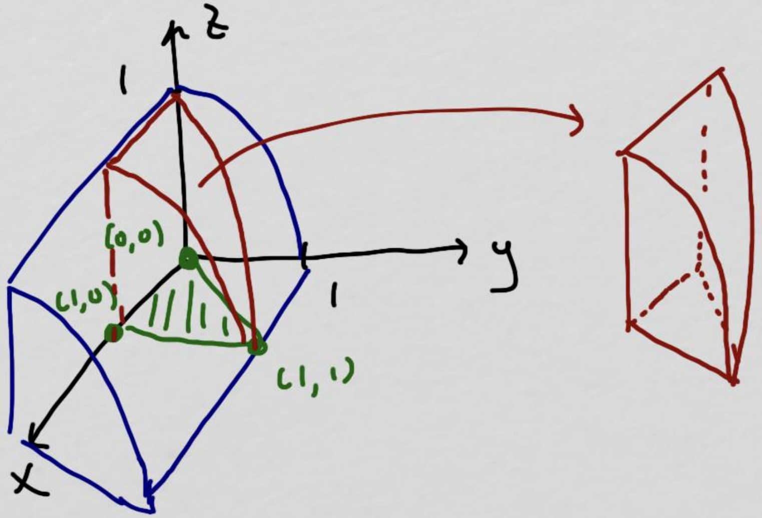
$$\int_0^1 y e^{x^2} \Big|_{y=x}^{y=1} dx = \int_0^1 \left(e^{x^2} - x e^{x^2}\right) dx$$

$$= \underbrace{\int_0^1 e^{x^2} dx}_{\text{still stuck}} - \underbrace{\int_0^1 x e^{x^2} dx}_{\text{by subs}}$$

so, we got a bit closer to the result but not all the way

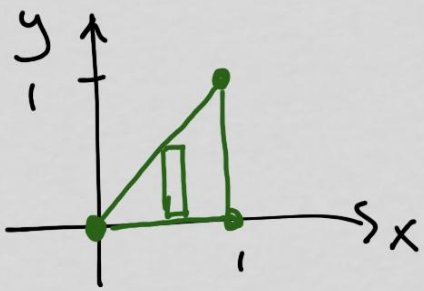
→ NOT all integrals can be done by hand

example Find the volume under $z = 1 - y^2$ above the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$



volume = ?

express triangle as Type II :



$0 \le x \le 1$

$0 \le y \le x$

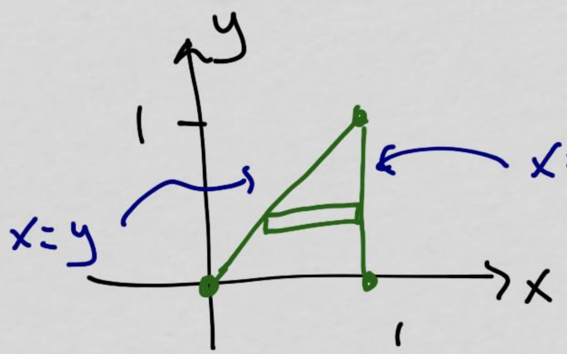
$$V = \int_0^1 \int_0^x (1-y^2)\, dy\, dx$$

integrate the height of surface : $z = f(x,y)$

$$= \int_0^1 y - \frac{1}{3}y^3 \Big|_{y=0}^{y=x} dx = \int_0^1 \left(x - \frac{1}{3}x^3\right) dx = \frac{1}{2}x^2 - \frac{1}{12}x^4 \Big|_0^1$$

$$= \boxed{\frac{5}{12}}$$

express the region as Type II



$0 \le y \le 1$

$y \le x \le 1$

$$V = \int_0^1 \int_y^1 (1-y^2)\, dx\, dy = \int_0^1 x - xy^2 \Big|_{x=y}^{x=1} dy$$

$$= \int_0^1 (1 - y^2 - y + y^3)\, dy = y - \frac{1}{3}y^3 - \frac{1}{2}y^2 + \frac{1}{4}y^4 \Big|_0^1$$

$$= \boxed{\frac{5}{12}}$$