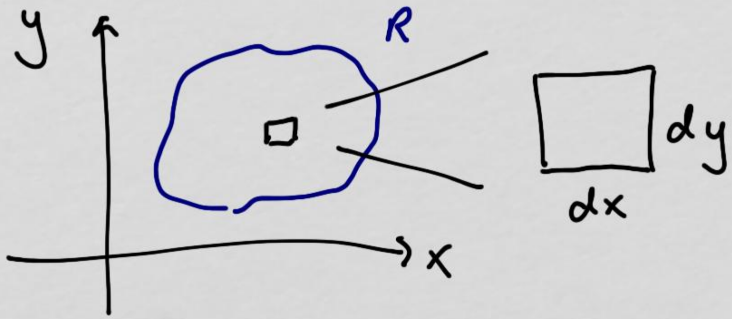
16.4 Triple Integrals

$\iint_R f(x,y)\,dA$ is the accumulation/integration of $f(x,y)$ on each area element dA all over region R

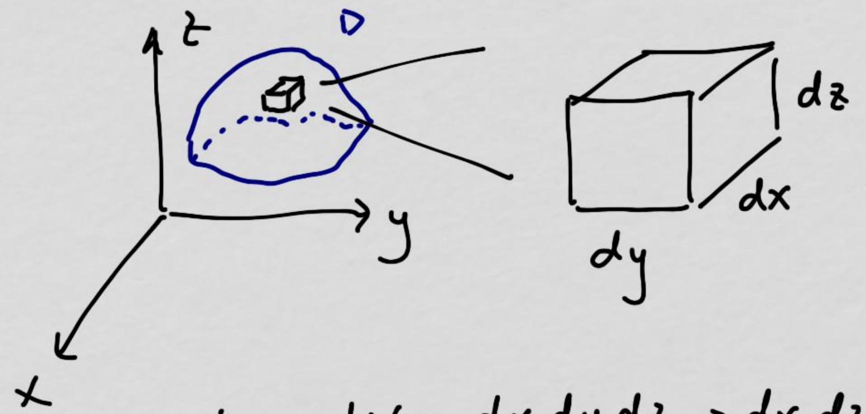


$dA = dx\,dy = dy\,dx$ two possible integration order

we accumulate $f(x,y)\,dA$ by integrating $\iint_R f(x,y)\,dA$

triple integral is essentially the same

$\iiint_D f(x,y,z)\,dV$ is accumulation of $f(x,y,z)$ on each <u>volume</u> element dV all over the volume D

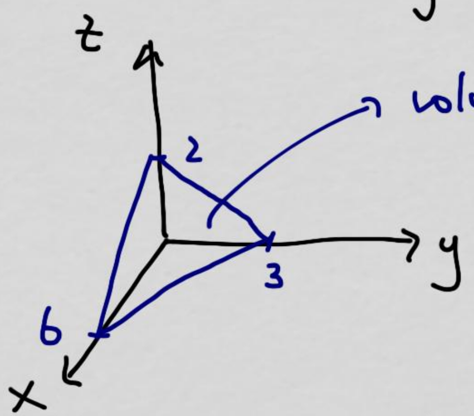


the volume of this is dV

note $dV = dx\,dy\,dz = dx\,dz\,dy$
$= dy\,dx\,dz = dy\,dz\,dx$
$= dz\,dx\,dy = dz\,dy\,dx$

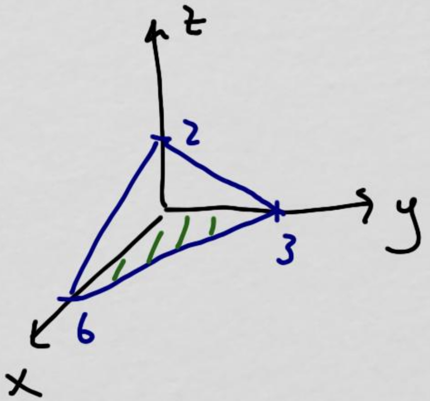
six possible ways
→ six possible integration order

<u>example</u> Use a triple integral to find the volume of the solid under $x+2y+3z=6$ in the first octant $(x>0, y>0, z>0)$

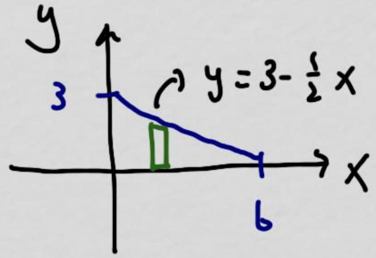


→ volume of this prism ?

volume is $\iiint\limits_D dV$ (just like area is $\iint\limits_R dA$)

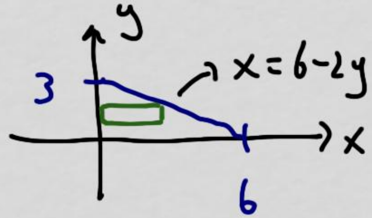


look at the projection onto xy-plane



Type I : $0 \le x \le 6$

$0 \le y \le 3 - \frac{1}{2} x$



Type II : $0 \le y \le 3$

$0 \le x \le 6 - 2y$

the "floor" of this prism is described by one of the two ways above

then on top of the "floor" the volume is as high as the plane

$\underbrace{x + 2y + 3z = 6}_{\text{solve for } z}$ allows us to go (and the volume is no lower than $z = 0$)

$$\Rightarrow \quad 0 \le z \le 2 - \frac{1}{3} x - \frac{2}{3} y$$

now, the three sets of bounds are:

$$0 \le x \le 6$$

$$0 \le y \le 3 - \tfrac{1}{2}x$$

$$0 \le z \le 2 - \tfrac{1}{3}x - \tfrac{2}{3}y$$

Basic rule: the variable with all constant bounds goes LAST (outside)

the variable with the most complicated bounds (involves the most number of variables) goes FIRST (inside)

so, here, x is LAST (outside), z is FIRST (inside), y is middle

$$\text{volume} = V = \underset{x}{\int_0^6} \underset{y}{\int_0^{3-\frac{1}{2}x}} \underset{z}{\int_0^{2-\frac{1}{3}x-\frac{2}{3}y}} \overbrace{dz\, dy\, dx}^{dV}$$

integrate inside-out as in double integrals

$$= \int_0^6 \int_0^{3-\frac{1}{2}x} z \Big|_{z=0}^{z=2-\frac{1}{3}x-\frac{2}{3}y} dy\,dx = \int_0^6 \int_0^{3-\frac{1}{2}x} \left(2-\frac{1}{3}x-\frac{2}{3}y\right) dy\,dx$$

$$= \int_0^6 2y - \frac{1}{3}xy - \frac{1}{3}y^2 \Big|_{y=0}^{y=3-\frac{1}{2}x} dx = \cdots = \boxed{6}$$

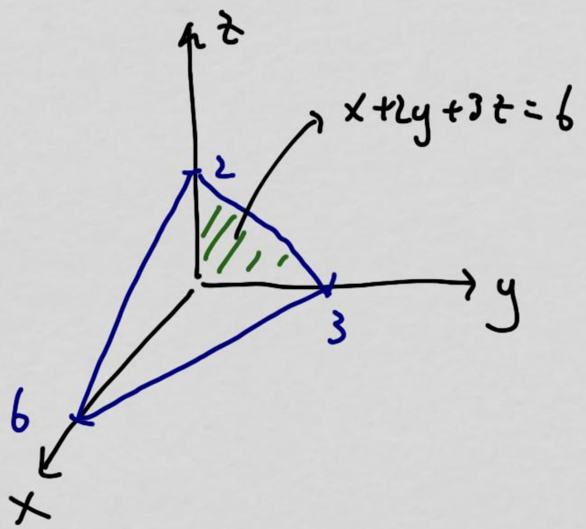
there are six possible orders, we just did with $dV = dz\,dy\,dx$

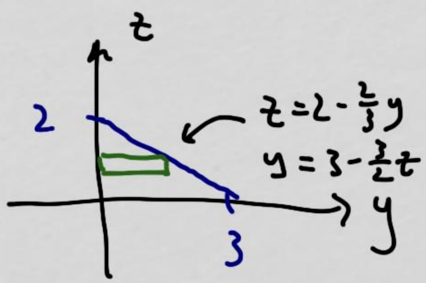
let's try another one: $dV = dx\,\underbrace{dy\,dz}_{}$

start with these two

this tells us what the "floor" plane is

here, the "floor" is the yz-plane





as a Type II (because z is last)

$0 \le z \le 2$

$0 \le y \le 3 - \frac{3}{2}z$

"floor" is yz-plane, the "ceiling" is in x-direction

x starts at $x=0$, up to as high as $x+2y+3z=6$ allows

↳ $x = 6-2y-3z$

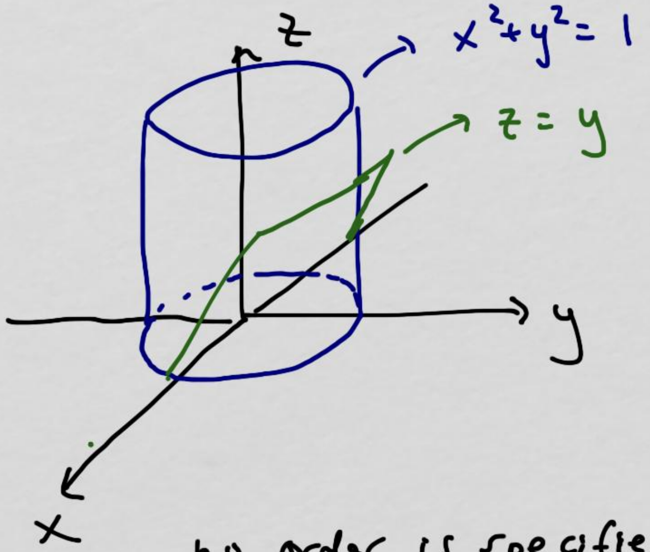
so, $0 \le x \le 6-2y-3z$

the integral is $$\int_0^2 \int_0^{3-\frac{3}{2}z} \int_0^{6-2y-3z} dx\,dy\,dz$$

$z \quad y \quad x$

$$= \int_0^2 \int_0^{3-\frac{3}{2}z} (6-2y-3z)\,dy\,dz = \cdots = 6$$

<u>example</u> Find the volume of the part of the cylinder $x^2+y^2=1$ bounded by the xy-plane and the plane $z=y$



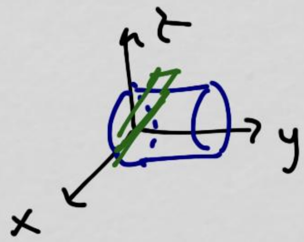
the part of $x^2+y^2=1$ cut off by $z=0$ and $z=y$ is

wedge shape

no order is specified, pick one

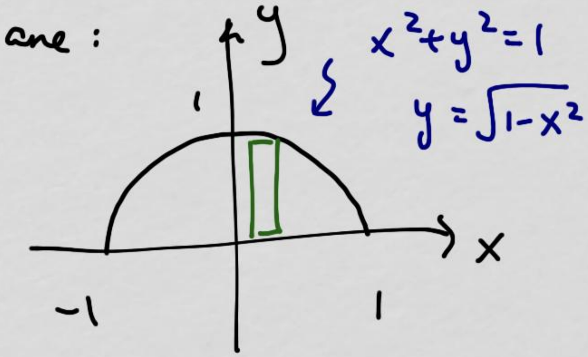
the cylinder has a simple projection onto xy-plane (circle), so xy-plane is a good choice as "floor" $\longrightarrow$ $dz\,dy\,dx$ or $dz\,dx\,dy$

if the shape were



then xz-plane is better

projection onto xy-plane:



only half because $z=y$ slices through x-axis

so, $-1 \le x \le 1$

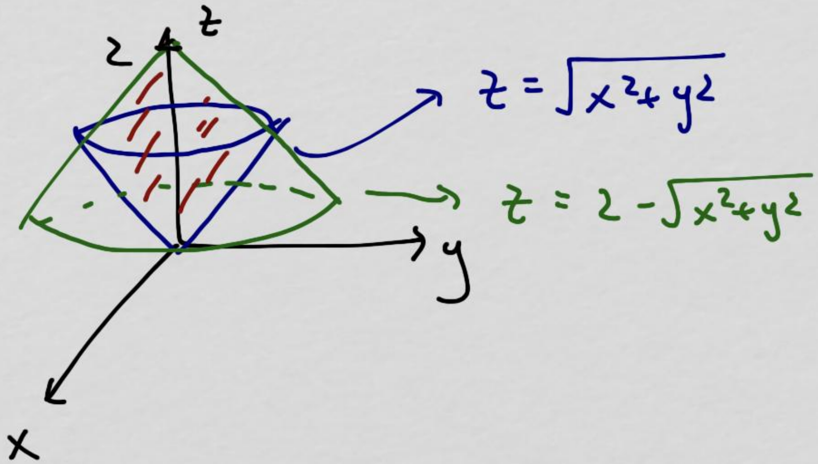
$0 \le y \le \sqrt{1-x^2}$

then the "ceiling" is bounded by $z=0$ and $z=y \Rightarrow 0 \le z \le y$

volume: $$\underset{x}{\int_{-1}^{1}} \underset{y}{\int_{0}^{\sqrt{1-x^2}}} \underset{z}{\int_{0}^{y}} dz\,dy\,dx = \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} z \Big|_{z=0}^{z=y} dy\,dx$$

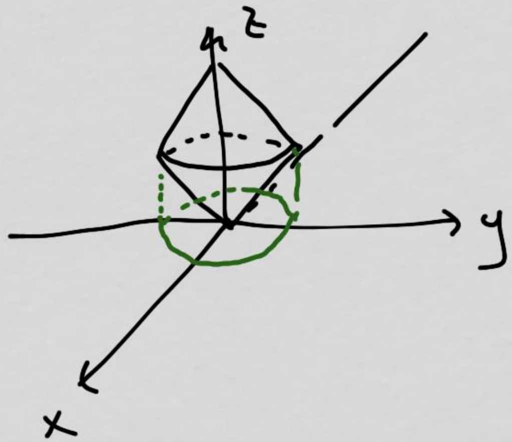
$$= \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} y\,dy\,dx = \int_{-1}^{1} \frac{1}{2} y^2 \Big|_{y=0}^{y=\sqrt{1-x^2}} dx = \cdots = \boxed{\frac{2}{3}}$$

<u>example</u> Volume above $z = \sqrt{x^2+y^2}$ and $z = 2-\sqrt{x^2+y^2}$



we want volume of this

no order specified, pick one: again, xy-plane is a natural "floor"



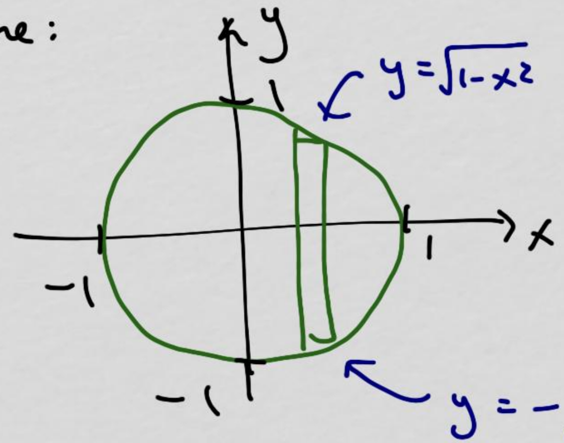
the projection is based on the intersection of the two cones → circle, but what radius?

$z = \sqrt{x^2+y^2}$ $\qquad$ $z = 2-\sqrt{x^2+y^2}$

intersects: $\sqrt{x^2+y^2} = 2-\sqrt{x^2+y^2}$

$$2\sqrt{x^2+y^2} = 2 \longrightarrow x^2+y^2 = 1 \quad \text{circle of radius } 1$$

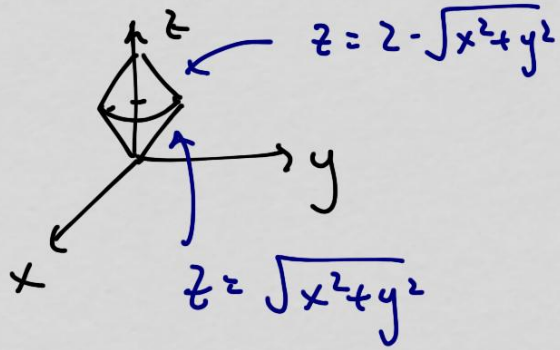
onto the xy-plane:



as Type I: $-1 \le x \le 1$

$$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$$

the ceiling is as high as the top cone allows and as low as the bottom cone allows



$$\sqrt{x^2+y^2} \le z \le 2-\sqrt{x^2+y^2}$$

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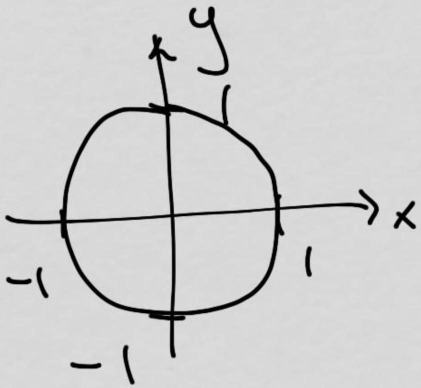
so, volume is $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{2-\sqrt{x^2+y^2}} dz\,dy\,dx$

$x \quad y \quad z$

this is a terrible integral in Cartesian. Since the "floor" is a circle, we expect the integral to be easier in Polar



in Polar, the "floor" is

$$0 \le \theta \le 2\pi$$

$$0 \le r \le 1$$

and don't forget: $dy\,dx = \underline{\underline{r}}\,dr\,d\theta$

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the integral $$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{2-\sqrt{x^2+y^2}} dz\,dy\,dx$$

$x \quad y \quad z$

becomes : $$\int_{0}^{2\pi}\int_{0}^{1}\int_{r}^{2-r} r\,dz\,dr\,d\theta \qquad x^2+y^2=r^2$$

$\theta \quad r \quad z$

$$\sqrt{x^2+y^2}=\sqrt{r^2}=r$$

$$= \int_{0}^{2\pi}\int_{0}^{1} rz \Big|_{z=r}^{z=2-r} dr\,d\theta$$

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$$= \int_0^{2\pi} \int_0^1 r(2-2r)\, dr\, d\theta = \int_0^{2\pi} \int_0^1 (2r-2r^2)\, dr\, d\theta$$

$$= \int_0^{2\pi} r^2 - \frac{2}{3} r^3 \Big|_0^1 d\theta = 2\pi \cdot \frac{1}{3} = \boxed{\frac{2\pi}{3}}$$

<u>example</u> Rewrite $\displaystyle\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dz\, dy\, dx$

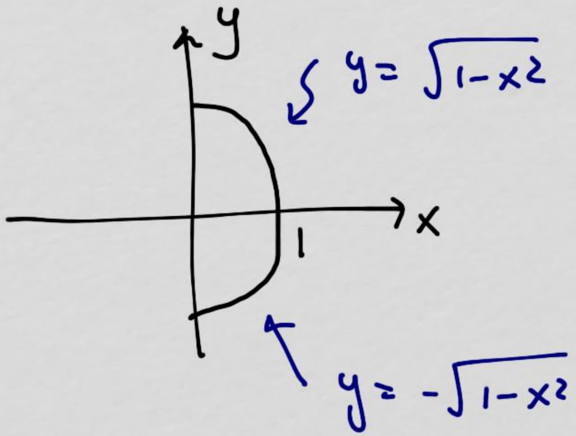
as $\displaystyle\iiint_D dx\, dz\, dy$ and evaluate.

as written:

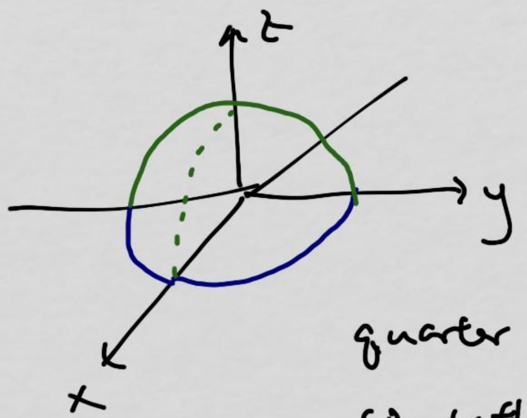
$$\left.\begin{aligned} 0 &\le x \le 1 \\ -\sqrt{1-x^2} &\le y \le \sqrt{1-x^2} \end{aligned}\right\} \text{"floor"}$$

$$0 \le z \le \sqrt{1-x^2-y^2} \quad \text{"ceiling"}$$

Sketch the solid:



"floor"



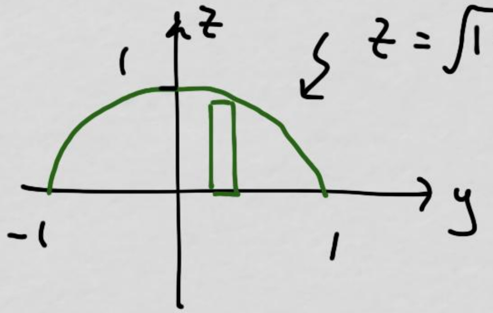
ceiling is $z = \sqrt{1-x^2-y^2}$

this is the upper hemisphere of a sphere radius 1

quarter of a sphere radius 1

so, without integrating, we know the volume is $\frac{1}{4} \cdot \frac{4}{3}\pi(1)^3 = \frac{\pi}{3}$

new order : $\underbrace{dx\,dz\,dy}_{\text{"floor"}}$



$z = \sqrt{1-x^2-y^2}$ w/ $x=0$

$z = \sqrt{1-y^2}$

$-1 \leq y \leq 1$

$0 \leq z \leq \sqrt{1-y^2}$

the "ceiling" is in x-direction : $0 \leq x \leq \underbrace{\sqrt{1-y^2-z^2}}$

same sphere equation expressed as $x = \ldots$

integral : $\displaystyle \int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} \int_{0}^{\sqrt{1-y^2-z^2}} dx\,dz\,dy = \frac{\pi}{3}$ (from sphere volume formula)

$y \quad z \quad x$